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On the Caudrey-Beals-Coifman System and the Gauge Group Action

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Abstract

The generalized Zakharov–Shabat systems with complex-valued Cartan elements and the systems studied A.V. Mikhailov, and later on by Caudrey, Beals and Coifman (CBC systems), and their gauge equivalent are studied. This includes: the properties of fundamental analytical solutions (FAS) for the gauge-equivalent to CBC systems and the minimal set of scattering data; the description of the class of nonlinear evolutionary equations solvable by the inverse scattering method and the recursion operator, related to such systems; the hierarchies of Hamiltonian structures.

1 Introduction

The idea that the inverse scattering method (ISM) is a generalized Fourier transform has appeared as early as 1974 in [1]. In the class of nonlinear evolution equations (NLEE) related to the Zakharov–Shabat (ZS) system [30, 28], the Lax operator belonging to $sl(2)$ algebra was studied. This class of NLEE contains such physically important equations as the nonlinear Schrödinger equation (NLS), the sine-Gordon and modified Korteveg–de-Vriez (mKdV) equations.

The multi-component ZS system leads to such important systems as the multi-component NLS, the N-wave type equations, etc.

Here, we consider the $n\times n$ system [5, 7, 11]:

$$L\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + q(x,t) - \lambda J \right) \Psi(x,t,\lambda), \tag{1.1}$$

where $q(x,t)$ and J take values in the semi-simple Lie algebra $\mathfrak{g}$ [25, 14, 29, 12]:

$$q(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha(x,t)E_\alpha + q_{-\alpha}(x,t)E_{-\alpha}) \in \mathfrak{g}_J \qquad J = \sum_{j=1}^{r} a_j H_j \in \mathfrak{h}.$$

For the case of complex J we will refer this system as Caudrey-Beals-Coifman (CBC) system. Here, J is a regular element in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathfrak{g}_J$ is the image of ad_J, $\{E_\alpha, H_i\}$ form the Cartan–Weyl Basis in $\mathfrak{g}$, Δ_+ is the set of positive roots of the algebra, $r = \operatorname{rank}\mathfrak{g} = \dim\mathfrak{h}$. For more details see section 2 below. The regularity of the Cartan elements means that $\mathfrak{g}_J$ is spanned by all root vectors E_α of $\mathfrak{g}$, i.e. $\alpha(J) \neq 0$ for any root α of $\mathfrak{g}$.

The given NLEE as well as the other members of its hierarchy possess Lax representation of the form (according to (1.1)): $[L(\lambda), M_P(\lambda)] = 0$, where

$$M_P\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + \sum_{k=-S}^{P-1} V_k(x,t) - \lambda^P f_P I \right) \Psi(x,t,\lambda) = 0, \qquad I \in \mathfrak{h}, \tag{1.2}$$

which must hold identically with respect to λ. A standard procedure generalizing the AKNS one [1] allows us to evaluate $V_k(x,t)$ in terms of $q(x,t)$ and its x-derivatives. Here and below, we consider only the class of potentials $q(x,t)$ vanishing fast enough for $|x| \to \infty$. Then one may also check that the asymptotic value of the potential in $M_P(\lambda)$ namely $f^{(P)}(\lambda) = f_P\lambda^P I$ may be understood as the dispersion law of the corresponding NLEE.

Another important trend in the development of IST was the introduction of the reduction group by A. V. Mikhailov [24], and further developed in [11, 12, 29, 25, 17, 15, 16, 23]. This allows one to prove that some of the well known models in the field theory [24] and also a number of new interesting NLEE [24, 11, 25] are integrable by the ISM and possess special symmetry properties. As a result its potential $q(x,t)$ has a very special form and J can no-longer be chosen real. The reduction group concept is important also because of the fact, that when one considers Lie-algebra-valued Lax operators, the number of independent fields grows rather quickly with the rank of the algebra: the corresponding NLEE are solvable for any rank, but their possible applications to physics do not seem realistic. However, one still may extract new integrable and physically useful NLEE by imposing reductions on L, i.e. algebraic restrictions on the potential of L, which diminish the number of independent functions in them and the number of equations [24]. Of course, such restrictions must be compatible with the dynamics of the NLEE.

The problem of constructing the spectral theory for (1.1) in the most general case when J has an arbitrary complex eigenvalues was initiated by Mikhailov [24], further developed by Beals, Coifman, Caudrey [2, 3, 4, 7] and continued by Zhou [31] in the case when the algebra $\mathfrak{g}$ is $sl(n)$, $q(x,t)$ vanishing fast enough for $|x| \to \infty$ and no a priori symmetry conditions are imposed on $q(x,t)$. This has been done later for any semi-simple Lie algebras by Gerdjikov and Yanovski [18].

The zero-curvature condition $[L(\lambda), M_P(\lambda)] = 0$, is invariant under the action of the group of gauge transformations [32]. Therefore the gauge equivalent systems are again completely integrable, possess a hierarchy of Hamiltonian structures, etc, [9, 28, 18, 32].

The structure of this paper is as follows: In section 2 we summarize some basic facts about the reduction group and Lie algebraic details. The construction of the fundamental analytic solutions (FAS) is sketched in section 3 which is done separately for the case of real Cartan elements (section 3.1) and for complex ones (section 3.2). The gauge equivalent NLEE's to the CBC systems are described in section 4.

2 Preliminaries

2.1 Simple Lie Algebras

Here, we fix up the notations and the normalization conditions for the Cartan-Weyl generators of $\mathfrak{g}$ [21]. We introduce $h_k \in \mathfrak{h}$, $k = 1, \ldots, r$ and E_α, $\alpha \in \Delta$ where $\{h_k\}$ are the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$. Along with h_k, we introduce also

$$H_\alpha = \frac{2}{(\alpha,\alpha)} \sum_{k=1}^{r} (\alpha, e_k) h_k, \quad \alpha \in \Delta, \tag{2.1}$$

where (α, e_k) is the scalar product in the root space $\mathbb{E}^r$ between the root α and e_k. The commutation relations are given by:

$$[h_k, E_\alpha] = (\alpha, e_k) E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha, \quad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\}. \end{cases}$$

We will denote by $\vec{a} = \sum_{k=1}^{r} a_k e_k$ the r-dimensional vector dual to $J \in \mathfrak{h}$; obviously $J = \sum_{k=1}^{r} a_k h_k$. If J is a regular real element in $\mathfrak{h}$ then without restrictions we may use it to introduce an ordering in Δ. Namely we will say that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, \vec{a}) > 0$ ($(\alpha, \vec{a}) < 0$ respectively). The normalization of the basis is determined by:

$$E_{-\alpha} = E_\alpha^T, \quad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha,\alpha)}, \quad N_{-\alpha,-\beta} = -N_{\alpha,\beta}, \quad N_{\alpha,\beta} = \pm(p+1), \tag{2.2}$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \dots, p$ $\alpha + (p+1)\beta \notin \Delta$ and $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}$. The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections A_α^*; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as $A_\alpha^* \vec{y} = \vec{y} - \frac{2(\alpha,\vec{y})}{(\alpha,\alpha)} \alpha, \quad \alpha \in \Delta$. All Weyl reflections A_α^* form a finite group $W_\mathfrak{g}$ known as the Weyl group. One may introduce in a natural way an action of the Weyl group on the Cartan-Weyl basis, namely:

$$A_\alpha^*(H_\beta) \equiv A_\alpha H_\beta A_\alpha^{-1} = H_{A_\alpha^* \beta}, \qquad A_\alpha^*(E_\beta) \equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{A_\alpha^* \beta}, \quad n_{\alpha,\beta} = \pm 1.$$

It is also well known that the matrices A_α are given (up to a factor from the Cartan subgroup) by $A_\alpha = \mathrm{e}^{E_\alpha} \mathrm{e}^{-E_{-\alpha}} \mathrm{e}^{E_\alpha} H_A$, where H_A is a conveniently chosen element from the Cartan subgroup such that $H_A^2 = \mathbb{1}$.

2.2 The Reduction Group

The main idea underlying Mikhailov's reduction group [24] is to impose algebraic restrictions on the Lax operators L and M which will be automatically compatible with the corresponding equations of motion. Due to the purely Lie-algebraic nature of the Lax representation this is most naturally done by imbedding the reduction group as a subgroup of $\mathrm{Aut}\,\mathfrak{g}$ – the group of automorphisms of $\mathfrak{g}$. Obviously, to each reduction imposed on L and M there will correspond a reduction of the space of fundamental solutions $\mathbf{S}_\Psi \equiv \{\Psi(x,t,\lambda)\}$ of (1.1).

Some of the simplest $\mathbb{Z}_2$-reductions of Zakharov–Shabat systems have been known for a long time (see [24]) and are related to outer automorphisms of $\mathfrak{g}$ and $\mathfrak{G}$, namely:

$$C_1(\Psi(x,t,\lambda)) = A_1 \Psi^\dagger(x,t,\kappa(\lambda)) A_1^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda), \qquad \kappa(\lambda) = \pm\lambda^*, \tag{2.3}$$

$$C_2(\Psi(x,t,\lambda)) = A_3 \Psi^*(x,t,\kappa(\lambda)) A_3^{-1} = \tilde{\Psi}(x,t,\lambda), \tag{2.4}$$

where A_1 and A_3 are elements of the group of automorphisms $\mathrm{Aut}\,\mathfrak{g}$ of the algebra $\mathfrak{g}$. Since our aim is to preserve the form of the Lax pair, we limit ourselves to automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. The reduction group G_R is a finite group which preserves the Lax representation, i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations:

i) $G_R \subset \mathrm{Aut}\mathfrak{g}$
ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane.
To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [24]:

$$C_k(U(\Gamma_k(\lambda))) = \eta_k U(\lambda), \tag{2.5}$$

where $U(x,\lambda) = q(x) - \lambda J$, $C_k \in \mathrm{Aut}\,\mathfrak{g}$ and $\Gamma_k(\lambda)$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

It is well known that $\mathrm{Aut}\,\mathfrak{g} \equiv V \otimes \mathrm{Aut}_0\mathfrak{g}$ where V is the group of outer automorphisms (the symmetry group of the Dynkin diagram) and $\mathrm{Aut}_0\mathfrak{g}$ is the group of inner automorphisms. Since we start with $I, J \in \mathfrak{h}$ it is natural to consider only those inner automorphisms that preserve the Cartan subalgebra $\mathfrak{h}$. Then $\mathrm{Aut}_0\mathfrak{g} \simeq \mathrm{Ad}_H \otimes W$ where Ad_H is the group of similarity transformations with elements from the Cartan subgroup and W is the Weyl group of $\mathfrak{g}$.

Generically each element $g_k \in G$ maps λ into a fraction-linear function of λ. Such action however is appropriate for a more general class of Lax operators which are linear fractional transformations of λ.

3 The Caudrey–Beals–Coifman systems

3.1 Fundamental analytical solutions and scattering data for real J.

The direct scattering problem for the Lax operator (1.1) is based on the Jost solutions:

$$\lim_{x\to\infty} \psi(x,\lambda)e^{i\lambda J x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda)e^{i\lambda J x} = \mathbb{1}, \tag{3.1}$$

and the scattering matrix

$$T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda). \tag{3.2}$$

The fundamental analytic solutions (FAS) $\chi^{\pm}(x,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\mathrm{Im}\,\lambda \gtrless 0$ and are related to the Jost solutions by [14]

$$\chi^{\pm}(x,\lambda) = \phi(x,\lambda)S^{\pm}(\lambda) = \psi^{\pm}(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda), \tag{3.3}$$

where $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ are the factors of the Gauss decomposition of the scattering matrix:

$$\begin{gathered} T(\lambda) = T^{-}(\lambda)D^{+}(\lambda)\hat{S}^{+}(\lambda) = T^{+}(\lambda)D^{-}(\lambda)\hat{S}^{-}(\lambda) \\ T^{\pm}(\lambda) = \exp\left(\sum_{\alpha>0} t^{\pm}_{\pm\alpha}(\lambda)E_{\alpha}\right), \qquad S^{\pm}(\lambda) = \exp\left(\sum_{\alpha>0} s^{\pm}_{\pm\alpha}(\lambda)E_{\alpha}\right), \\ D^{+}(\lambda) = I\exp\left(\sum_{j=1}^{r} \frac{2d^{+}(\lambda)}{(\alpha_j,\alpha_j)}H_j\right), \qquad D^{-}(\lambda) = I\exp\left(\sum_{j=1}^{r} \frac{2d^{-}(\lambda)}{(\alpha_j,\alpha_j)}H_j^{-}\right). \end{gathered} \tag{3.4}$$

Here $H_j = H_{\alpha_j}$, $H_j^{-} = w_0(H_j)$, $\hat{S} \equiv S^{-1}$, I is an element from the universal center of the corresponding Lie group $\mathfrak{G}$ and the superscript $+$ (or $-$) in the Gauss factors means upper- (or lower-) triangularity for $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$ and shows that $D^{+}(\lambda)$ (or $D^{-}(\lambda)$) are analytic functions with respect to λ for $\mathrm{Im}\,\lambda > 0$ (or $\mathrm{Im}\,\lambda < 0$ respectively).

On the real axis $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$ are linearly related by:

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda), \qquad G_0(\lambda) = S^+(\lambda)\hat{S}^-(\lambda), \tag{3.5}$$

and the sewing function $G_0(\lambda)$ may be considered as a minimal system of scattering data provided the Lax operator (1.1) has no discrete eigenvalues [14].

3.2 The CBC Construction for Semisimple Lie Algebras

Here we will sketch the construction of the FAS for the case of complex-valued regular Cartan element J: $\alpha(J) \neq 0$, following the general ideas of Beals and Coifman [2] for the $sl(n)$ algebras and [18] for the orthogonal and symplectic algebras. These ideas consist of the following:

1. For potentials $q(x)$ with small norm $||q(x)||_{L^1} < 1$, one can divide the complex λ–plane into sectors and then construct an unique FAS $m_\nu(x,\lambda)$ which is analytic in each of these sectors Ω_ν;

2. For these FAS in each sector there is a certain Gauss decomposition problem for the scattering matrix $T(\lambda)$ which has an unique solution in the case of absence of discrete eigenvalues.

The main difference between the cases of real-valued and complex-valued J lies in the fact that for complex J the Jost solutions and the scattering data exist only for the potentials on compact support. For potentials not on a compact support some additional conditions on the potential should be imposed [24].

We define the regions (sectors) Ω_ν as consisting of those λ's for which $\mathrm{Im}\,(\lambda\alpha(J)) \neq 0$ for any $\alpha \in \Delta$. Thus the boundaries of the Ω_ν's consist of the set of straight lines:

$$l_\alpha \equiv \{\lambda : \mathrm{Im}\,\lambda\alpha(J) = 0, \qquad \alpha \in \Delta\}, \tag{3.6}$$

and to each root α we can associate a certain line l_α; different roots may define coinciding lines.

Note that with the change from λ to $\lambda e^{i\eta}$ and J to $Je^{-i\eta}$ (this leads that the product $\lambda\alpha(J)$ invariant) we can always choose l_1 to be along the positive real λ axis.

To introduce an ordering in each sector Ω_ν we choose the vector $\vec{a}_\nu(\lambda) \in \mathbb{E}^r$ to be dual to the element $\mathrm{Im}\,\lambda J \in \mathfrak{h}$. Then in each sector we split Δ into

$$\Delta = \Delta_\nu^+ \cup \Delta_\nu^-, \qquad \Delta_\nu^\pm = \{\alpha \in \Delta : \mathrm{Im}\,\lambda\alpha(J) \gtrless 0,\, \lambda \in \Omega_\nu\}. \tag{3.7}$$

If $\lambda \in \Omega_\nu$ then $-\lambda \in \Omega_{M+\nu}$ (if the lines l_α split the complex λ-plane into $2M$ sectors). We need also the subset of roots:

$$\delta_\nu = \{\alpha \in \Delta : \mathrm{Im}\,\lambda\alpha(J) = 0,\, \lambda \in l_\nu\} \tag{3.8}$$

which will be a root system of some subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$. Then we can write that

$$\mathfrak{g} = \bigoplus_{\nu=1}^{M} \mathfrak{g}_\nu \qquad \Delta = \bigcup_{\nu=1}^{M} \delta_\nu \qquad \delta_\nu = \delta_\nu^+ \cup \delta_\nu^-, \qquad \delta_\nu^\pm = \delta_\nu \cap \Delta_\nu^\pm .$$

Thus we can describe in more details the sets $\Delta_\nu^\pm$:

$$\Delta_k^+ = \delta_1^+ \cup \delta_2^+ \cup \cdots \cup \delta_k^+ \cup \delta_{k+1}^- \cup \cdots \cup \delta_M^-, \quad \Delta_{k+M}^+ = \Delta_k^-, \quad k = 1,\ldots,M. \tag{3.9}$$

Note that each ordering in Δ can be obtained from the "canonical" one by an action of a properly chosen element of the weyl group $\mathfrak{W}(\mathfrak{g})$.

Now in each sector Ω_ν we introduce the FAS $\chi_\nu(x,\lambda)$ and $m_\nu(x,\lambda)=\chi_\nu(x,\lambda)e^{i\lambda Jx}$ satisfying the equivalent equation:

$$\mathrm{i}\frac{\mathrm{d}m_\nu}{\mathrm{d}x}+q(x)m_\nu(x,\lambda)-\lambda[J,m_\nu(x,\lambda)]=0, \qquad \lambda\in\Omega_\nu. \tag{3.10}$$

If $q(x)$ is a potential on compact support then the FAS $m_\nu(x,\lambda)$ are related to the Jost solutions by

$$\begin{aligned} m_\nu(x,\lambda)&=\phi(x,\lambda)S_\nu^+(\lambda)e^{i\lambda Jx}=\psi(x,\lambda)T_\nu^-(x,\lambda)D_\nu^+(\lambda)e^{i\lambda Jx},\\ m_{\nu-1}(x,\lambda)&=\phi(x,\lambda)S_\nu^-(\lambda)e^{i\lambda Jx}=\psi(x,\lambda)T_\nu^+(x,\lambda)D_\nu^-(\lambda)e^{i\lambda Jx}, \qquad \lambda\in l_\nu. \end{aligned} \tag{3.11}$$

From the definitions of $m_\nu(x,\lambda)$ and the scattering matrix $T(\lambda)$ we have

$$T(\lambda)=T_\nu^-(\lambda)D_\nu^+(\lambda)\hat{S}_\nu^+(\lambda)=T_\nu^+(\lambda)D_\nu^-(\lambda)\hat{S}_\nu^-(\lambda), \quad \lambda\in l_\nu \tag{3.12}$$

where in the first equality we take $\lambda=\mu e^{i0}$ and for the second– $\lambda=\mu e^{-i0}$ with $\mu\in l_\nu$. The corresponding expressions for the Gauss factors have the form:

$$\begin{aligned} S_\nu^+(\lambda)&=\exp\left(\sum_{\alpha\in\Delta_\nu^+}s_{\nu,\alpha}^+(\lambda)E_\alpha\right), \qquad S_\nu^-(\lambda)=\exp\left(\sum_{\alpha\in\Delta_{\nu-1}^+}s_{\nu,\alpha}^-(\lambda)E_{-\alpha}\right),\\ T_\nu^+(\lambda)&=\exp\left(\sum_{\alpha\in\Delta_{\nu-1}^+}t_{\nu,\alpha}^+(\lambda)E_\alpha\right), \qquad T_\nu^-(\lambda)=\exp\left(\sum_{\alpha\in\Delta_\nu^+}t_{\nu,\alpha}^-(\lambda)E_{-\alpha}\right),\\ D_\nu^+(\lambda)&=\exp(\mathbf{d}_\nu^+(\lambda)\cdot\mathbf{H}_\nu), \qquad D_\nu^-(\lambda)=\exp(\mathbf{d}_\nu^-(\lambda)\cdot\mathbf{H}_{\nu-1}). \end{aligned} \tag{3.13}$$

Here $\mathbf{d}_\nu^\pm(\lambda)=(d_{\nu,1}^\pm,\dots,d_{\nu,r}^\pm)$ is a vector in the root space and

$$\mathbf{H}_\eta=\left(\frac{2H_{\eta,1}}{(\alpha_{\eta,1},\alpha_{\eta,1})},\dots,\frac{2H_{\eta,r}}{(\alpha_{\eta,r},\alpha_{\eta,r})}\right), \qquad (\mathbf{d}_\nu^\pm(\lambda),\mathbf{H}_\eta)=\sum_{k=1}^r\frac{2d_{\nu,k}^\pm(\lambda)H_{\eta,k}}{(\alpha_{\eta,k},\alpha_{\eta,k})}, \tag{3.14}$$

where $\alpha_{\eta,k}$ is the k-th simple root of $\mathfrak{g}$ with respect to the ordering Δ_η^+ and $H_{\eta,k}$ are their dual elements in the Cartan subalgebra $\mathfrak{h}$.

4 The Gauge Group Action

4.1 The class of the gauge equivalent NLEEs

The notion of gauge equivalence allows one to associate to any Lax pair of the type (1.1), (1.2) an equivalent one [18], solvable by the inverse scattering method for the gauge equivalent linear problem:

$$\tilde{L}\tilde{\psi}\equiv\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}-\lambda S\right)\tilde{\psi}(x,t,\lambda)=0, \quad \tilde{M}\tilde{\psi}\equiv\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}-\lambda f(S)\right)\tilde{\psi}(x,t,\lambda)=0, \tag{4.1}$$

where $\tilde{\psi}(x,t,\lambda)=g^{-1}(x,t)\psi(x,t,\lambda)$,

$$S=\mathrm{Ad}_g\cdot J\equiv g^{-1}(x,t)Jg(x,t), \tag{4.2}$$

and $g(x,t)=m_\nu(x,t,0)$ is FAS at $\lambda=0$. The functions $m_\nu(x,t,\lambda)$ are analytic with respect to λ in each sector Ω_ν (in the case of potential on compact support). On the continuous spectrum of

L the FAS $m_\nu(x,t,0)$ and the Jost solutions $\psi(x,t,0)$ are linearly dependant (3.11): $m_\nu(x,t,0) = \psi(x,t,0)T_\nu^-(t,0)D_\nu^+(0)$. Therefore the gauge group action is well defined. The zero-curvature condition $[\tilde{L},\tilde{M}] = 0$ gives:

$$S_t - \frac{\mathrm{d}}{\mathrm{d}x} f(S) = 0, \tag{4.3}$$

where $f(S) = \sum_{p=0}^{r-1} \alpha_p S^{2p+1}$ is an odd polynomial of S. Both Lax operators $L(\lambda)$ and $\tilde{L}(\lambda)$ have equivalent spectral properties and spectral data and therefore the classes of NLEE's related to them are equivalent. It is natural that $f(\mathscr{S}) = g^{-1}(x,t)Ig(x,t)$, i.e., it is uniquely determined by I. Both J and I belong to the Cartan subalgebra $\mathfrak{h}$ so they have common set of eigenspaces.

1) $\mathfrak{g} \simeq \mathbf{A}_r = sl(n)$ with $n = r+1$. We have $J = \operatorname{diag}(J_1,\dots,J_n)$, $I = \operatorname{diag}(I_1,\dots,I_n)$, and the only constraint on the eigenvalues J_k and I_k is $\operatorname{tr} J = \operatorname{tr} I = 0$. The projectors on the common eigensubspaces of J and I are given by:

$$\pi_k(J) = \prod_{s\neq k} \frac{J - J_s}{J_k - J_s} = \operatorname{diag}(0,\dots,0,\underset{k}{1},0,\dots,0). \tag{4.4}$$

Next we note that $I = \sum_{k=1}^n I_k \pi_k(J)$. In order to derive $f(\mathscr{S})$ for $\mathfrak{g} \simeq sl(n)$ we need to apply the gauge transformation to (4.3) with the result: $f(\mathscr{S}) = \sum_{k=1}^n I_k \pi_k(\mathscr{S})$, i.e., $f(\mathscr{S})$ is a polynomial of order $n-1$. Obviously $\mathscr{S}$ is restricted by: $\prod_{k=1}^n (\mathscr{S} - J_k) = 0$, $\operatorname{tr}\mathscr{S}^k = \operatorname{tr} J^k$. for $k = 2,\dots,n$.

2) $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$ In order to express $f(S)$ through their eigenvalues J_k and I_k we introduce the diagonal matrix-valued functions:

$$f_k(J) = \frac{J}{J_k} \prod_{s\neq k} \frac{J^2 - J_s^2}{J_k^2 - J_s^2} = H_{e_k} \in \mathfrak{h}, \tag{4.5}$$

where by H_{e_k} we denote the element in $\mathfrak{h}$ dual to the basis vector e_k in the root space of $\mathfrak{g}$. Using (4.5) and applying Ad_g we get:

$$I = \sum_{k=1}^r I_k f_k(J), \qquad f(S) \equiv g^{-1}(x,t)Ig(x,t) = \sum_{k=1}^r I_k f_k(S). \tag{4.6}$$

In addition $S(x,t)$ satisfies the characteristic equations:

$$S^{\kappa_0} \prod_{k=1}^r (S^2 - J_k^2) = 0, \tag{4.7}$$

where $\kappa_0 = 0$ if $\mathfrak{g} \simeq C_r$ or D_r and $\kappa_0 = 1$, if $\mathfrak{g} \simeq B_r$.

Then the equation gauge equivalent to (1.1) becomes:

$$S_t - \alpha_0 S_x - \sum_{p=1}^{r-1} \alpha_p (S^{2p+1})_x = 0. \tag{4.8}$$

The function $S(x,t) \in \mathfrak{g}$ is also subject to constraints; one of them is provided by (4.7). To construct the others we assume that $\mathfrak{g} \simeq \mathbf{B}_r$ or $\mathbf{D}_r$ and use the typical representation of $\mathfrak{g}$. It this settings we easily see that all odd powers of H_{e_k} also belong to the Cartan subalgebra $\mathfrak{h}$. Thus we conclude that all odd powers of S also belong to $\mathfrak{g}$. The invariance properties of the trace lead to: $\operatorname{tr}(J^{2k}) \equiv 2\sum_{k=1}^r J_k^{2k} = \operatorname{tr}(S)^{2k}$, for $k = 1,\dots,r$. These are precisely r independent algebraic constraints on S. Solving for them we conclude that the number of independent coefficients in S is equal to the number of roots $|\Delta|$ of $\mathfrak{g}$.

4.2 The Minimal Set of Scattering Data for $L(\lambda)$ and $\tilde{L}(\lambda)$

We skip the details about CBC construction which can be found in [18] and go to the minimal set of scattering data for the case of complex J which are defined by the sets $\mathscr{F}_1$ and $\mathscr{F}_2$ as follows:

$$\mathscr{F}_1 = \bigcup_{\nu=1}^{2M} \mathscr{F}_{1,\nu}, \qquad \mathscr{F}_2 = \bigcup_{\nu=1}^{2M} \mathscr{F}_{2,\nu},$$
$$\mathscr{F}_{1,\nu} = \{\rho^{\pm}_{B,\nu,\alpha}(\lambda),\, \alpha \in \delta^+_\nu,\, \lambda \in l_\nu\} \qquad \mathscr{F}_{2,\nu} = \{\tau^{\pm}_{B,\nu,\alpha}(\lambda),\, \alpha \in \delta^+_\nu,\, \lambda \in l_\nu\}, \tag{4.9}$$

where

$$\rho^{\pm}_{B,\nu,\alpha}(\lambda) = \langle S^{\pm}_\nu(\lambda) B \hat{S}^{\pm}_\nu(\lambda), E_{\mp\alpha}\rangle, \qquad \tau^{\pm}_{B,\nu,\alpha}(\lambda) = \langle T^{\pm}_\nu(\lambda) B \hat{T}^{\pm}_\nu(\lambda), E_{\mp\alpha}\rangle, \tag{4.10}$$

with $\alpha \in \delta^+_\nu$, $\lambda \in l_\nu$ and B is a properly chosen regular element of the Cartan subalgebra $\mathfrak{h}$. Without loss of generality we can take in (4.10) $B = H_\alpha$. Note that the functions $\rho^{\pm}_{B,\nu,\alpha}(\lambda)$ and $\tau^{\pm}_{B,\nu,\alpha}(\lambda)$ are continuous functions of λ for $\lambda \in l_\nu$.

If we choose J in such way that $2M = |\Delta|$– the number of the roots of $\mathfrak{g}$. then to each pair of roots $\{\alpha, -\alpha\}$ one can relate a separate pair of rays $\{l_\alpha, l_{\alpha+M}\}$, and $l_\alpha \neq l_\beta$ if $\alpha \neq \pm\beta$. In this case each of the subalgebras $\mathfrak{g}_\alpha$ will be isomorphic to $sl(2)$.

In order to determine the scattering data for the gauge equivalent equations we need to start with the FAS for these systems:

$$\tilde{m}^{\pm}_\nu(x,\lambda) = g^{-1}(x,t) m^{\pm}_\nu(x,\lambda) g_-, \tag{4.11}$$

where $g_- = \lim_{x\to-\infty} g(x,t)$ and due to (1.2) and $g_- = \hat{T}(0)$. In order to ensure that the functions $\tilde{\xi}^{\pm}(x,\lambda)$ are analytic with respect to λ the scattering matrix $T(0)$ at $\lambda = 0$ must belong to the corresponding Cartan subgroup $\mathfrak{H}$. Then Equation (4.11) provide the fundamental analytic solutions of $\tilde{L}$. We can calculate their asymptotics for $x \to \pm\infty$ and thus establish the relations between the scattering matrices of the two systems:

$$\lim_{x\to-\infty} \tilde{\xi}^{+}(x,\lambda) = e^{-\mathrm{i}\lambda Jx} T(0) S^{+}(\lambda) \hat{T}(0) \qquad \lim_{x\to\infty} \tilde{\xi}^{+}(x,\lambda) = e^{-\mathrm{i}\lambda Jx} T^{-}(\lambda) D^{+}(\lambda) \hat{T}(0) \tag{4.12}$$

with the result: $\tilde{T}(\lambda) = T(\lambda)\hat{T}(0)$. Obviously $\tilde{T}(0) = \mathbb{1}$. The factors in the corresponding Gauss decompositions are related by:

$$\tilde{S}^{\pm}(\lambda) = T(0) S^{\pm}(\lambda) \hat{T}(0), \qquad \tilde{T}^{\pm}(\lambda) = T^{\pm}(\lambda) \qquad \tilde{D}^{\pm}(\lambda) = D^{\pm}(\lambda)\hat{T}(0).$$

On the real axis again the FAS $\tilde{\xi}^{+}(x,\lambda)$ and $\tilde{\xi}^{-}(x,\lambda)$ are related by $\tilde{\xi}^{+}(x,\lambda) = \tilde{\xi}^{-}(x,\lambda)\tilde{G}_0(\lambda)$ with the normalization condition $\tilde{\xi}(x,\lambda=0) = \mathbb{1}$ and $\tilde{G}_0(\lambda) = \tilde{S}^{+}(\lambda)\hat{\tilde{S}}^{-}(\lambda)$ again can be considered as a minimal set of scattering data.

The minimal set of scattering data for the gauge-equivalent CBC systems are defined by the sets $\tilde{\mathscr{F}}_1$ and $\tilde{\mathscr{F}}_2$ as follows:

$$\tilde{\mathscr{F}}_1 = \bigcup_{\nu=1}^{2M} \tilde{\mathscr{F}}_{1,\nu}, \qquad \tilde{\mathscr{F}}_2 = \bigcup_{\nu=1}^{2M} \tilde{\mathscr{F}}_{2,\nu},$$
$$\tilde{\mathscr{F}}_{1,\nu} = \{\tilde{\rho}^{\pm}_{B,\nu,\alpha}(\lambda),\, \alpha \in \delta^+_\nu,\, \lambda \in l_\nu\} \qquad \tilde{\mathscr{F}}_{2,\nu} = \{\tilde{\tau}^{\pm}_{B,\nu,\alpha}(\lambda),\, \alpha \in \delta^+_\nu,\, \lambda \in l_\nu\}, \tag{4.13}$$

where

$$\tilde{\rho}^{\pm}_{B,\nu,\alpha}(\lambda) = \langle T(0) S^{\pm}_\nu(\lambda) B \hat{S}^{\pm}_\nu(\lambda) \hat{T}(0), E_{\mp\alpha}\rangle, \quad \tilde{\tau}^{\pm}_{B,\nu,\alpha}(\lambda) = \langle T^{\pm}_\nu(\lambda) B \hat{T}^{\pm}_\nu(\lambda), E_{\mp\alpha}\rangle, \tag{4.14}$$

with $\alpha \in \delta^+_\nu$, $\lambda \in l_\nu$ and B is again a properly chosen regular element of the Cartan subalgebra $\mathfrak{h}$. Without loss of generality we can take in (4.14) $B = H_\alpha$ (as in (4.10)). That the functions $\tilde{\rho}^{\pm}_{B,\nu,\alpha}(\lambda)$ and $\tilde{\tau}^{\pm}_{B,\nu,\alpha}(\lambda)$ are continuous functions of λ for $\lambda \in l_\nu$ and have the same analyticity properties as the functions $\rho^{\pm}_{B,\nu,\alpha}(\lambda)$ and $\tau^{\pm}_{B,\nu,\alpha}(\lambda)$.

4.3 Integrals of Motion and Hierarchies of Hamiltonian Structures

Both classes of NLEE's are infinite dimensional completely integrable Hamiltonian systems and possess hierarchies of Hamiltonian structures.

The phase space $\mathscr{M}_{\rm CBC}$ is the linear space of all off-diagonal matrices $q(x,t)$ tending fast enough to zero for $x \to \pm\infty$. The hierarchy of pair-wise compatible symplectic structures on $\mathscr{M}_{\rm CBC}$ is provided by the 2-forms:

$$\Omega^{(k)}_{\rm CBC} = {\rm i}\int_{-\infty}^{\infty} dx {\rm tr}\left(\delta q(x,t)\wedge\Lambda^k[J,\delta q(x,t)]\right), \tag{4.15}$$

where $\Lambda = (\Lambda_+ + \Lambda_-)/2$ is the generating (recursion) operator for (1.1) defined as follows:

$$\Lambda_\pm Z(x) = {\rm ad}_J^{-1}(1-\pi_0)\left(i\frac{dZ}{dx} + [q(x),Z(x)] + i\left[q(x),\pi_0\int_{\pm\infty}^{x} dy\,[q(y),Z(y)]\right]\right),$$

where $\pi_0(X) = {\rm ad}_J^{-1}\circ{\rm ad}_J(X)$. The symplectic forms $\Omega^{(k)}_{\rm CBC}$ can be expressed in terms of the scattering data for $L(\lambda)$:

$$\begin{aligned}
\Omega^{(k)}_{\rm CBC} &= \frac{c_k}{2\pi}\sum_{\nu=1}^{M}\int_{\lambda\in l_\nu\cup l_{M+\nu}} d\lambda\,\lambda^k\left(\Omega^+_{0,\nu}(\lambda) - \Omega^-_{0,\nu}(\lambda)\right),\\
\Omega^\pm_{0,\nu}(\lambda) &= \left\langle \hat{D}^\pm_\nu(\lambda)\hat{T}^\mp_\nu(\lambda)\delta T^\mp_\nu(\lambda) D^\pm_\nu(\lambda)\wedge\hat{S}^\pm_\nu(\lambda)\delta S^\pm_\nu(\lambda)\right\rangle.
\end{aligned} \tag{4.16}$$

Note that the kernels of $\Omega^{(k)}_{\rm CBC}$ differ only by the factor λ^k so all of them can be casted into canonical form simultaneously.

The phase space $\mathscr{M}_{\rm gauge}$ of the gauge equivalent to the CBC systems is the manifold of all $\mathscr{S}(x,t)$ determined by the second relation in (4.2). The family of compatible 2-forms is:

$$\tilde{\Omega}^{(k)}_{\rm gauge} = \frac{i}{4}\int_{-\infty}^{\infty} dx {\rm tr}\left(\delta S^{(0)}\wedge\tilde{\Lambda}^k[S^{(0)},\delta S^{(0)}(x,t)]\right). \tag{4.17}$$

Here $\tilde{\Lambda}$ is the recursion operator for the gauge equivalent to the CBC systems:

$$\widetilde{\Lambda}_\pm\widetilde{Z} = i{\rm ad}^{-1}_{S(x)}(1-\widetilde{\pi}_0(x))\left\{\frac{d\widetilde{Z}}{dx} + \sum_{k=1}^{2}[\widetilde{h}_k(x),{\rm ad}^{-1}_{\mathscr{S}(x)}]\int_{\pm\infty}^{x} dy\left\langle[\widetilde{h}_k(y),{\rm ad}^{-1}_{\mathscr{S}(y)}\mathscr{S}_y],\widetilde{Z}(y)\right\rangle,\right\}$$

where $\widetilde{h}_k(x,t) = g^{-1}(x,t)H_k g(x,t)$, and $\langle H_k,H_j\rangle = \langle\widetilde{h}_k(x,t),\widetilde{h}_j(x,t)\rangle = \delta_{jk}$.

The spectral theory of these two operators Λ and $\tilde{\Lambda}$ underlie all the fundamental properties of these two classes of gauge equivalent NLEE, for details see [18]. Note that the gauge transformation relates nontrivially the symplectic structures, i.e. $\Omega^{(k)}_{\rm NLSE}\simeq\tilde{\Omega}^{(k+2)}_{\rm HFE}$ [26, 18].

5 Conclusions

In the present article the gauge-equivalent models, related to the Lax pair (4.1) are studied and their relations to the "canonical" systems (1.1), (1.2) are established. This includes: the description of the class of nonlinear evolutionary equations solvable by the inverse scattering method (Section 4.1), the minimal set of scattering data (Section 4.2); the integrals of motion and the hierarchies of Hamiltonian structures (Section 4.3).

We will finish this article with several concluding remarks. To CBC systems and their gauge equivalent one can apply the analysis [18] and derive the completeness relations for the corresponding system of squared solutions. Such analysis will allow one to prove the pair-wise compatibility of the Hamiltonian structures and eventually to derive their action-angle variables, see e.g. [27] and [5] for the $\mathbf{A}_r$-series.

For the case of singular J ($\alpha(J) = 0$) the construction of FAS $m_\nu(x,t,\lambda)$ and $\tilde{m}_\nu(x,t,\lambda)$ requires the use of generalized Gauss decomposition in which the factors $D_\nu^\pm(\lambda)$ are block-diagonal, while $T_\nu^\pm(t,\lambda)$ and $S_\nu^\pm(t,\lambda)$ are block-triangular. This will be addressed to a subsequent paper.

The approach presented here allows one to consider CBC systems with more general λ- dependence, like the Principal Chiral field models and other relativistic invariant fiels theories [29].

If $\mathfrak{g} \simeq so(5)$ then the corresponding gauge equivalent system describes isoparametric surfaces [10].

Finally, some open problems are:
1) to study the internal structure of the soliton solutions and soliton interactions (for both types of systems);
2) to study reductions of the gauge equivalent to CBC systems.

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