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A Common Integrable Structure in the Hermitian Matrix Model and Hele-Shaw Flows

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Abstract

It is proved that the system of string equations of the dispersionless 2-Toda hierarchy which arises in the planar limit of the hermitian matrix model also underlies certain processes in Hele-Shaw flows.

1 Introduction

The Toda hierarchy represents a relevant integrable structure which emerges in several random matrix models [1]-[3]. Thus, the partition functions

$$Z_N(\text{Hermitian}) = \int \mathrm{d}H \exp\Big(\mathrm{tr}(\sum_{k\geqslant 1} t_k H^k)\Big), \tag{1.1}$$

$$Z_N(\text{Normal}) = \int \mathrm{d}M\, \mathrm{d}M^\dagger \exp\Big(\mathrm{tr}\,(M M^\dagger + \sum_{k\geqslant 1}(t_k M^k + \bar{t}_k M^{\dagger k}))\Big), \tag{1.2}$$

of the hermitian ($H = H^\dagger$) and the normal matrix models ($[M, M^\dagger] = 0$) , where N is the matrix dimension, are tau-functions of the 1-Toda and 2-Toda hierarchy, respectively. As a consequence of this connection new facets of the Toda hierarchy have been discovered. Thus the analysis of the large N-limit of the Hermitian matrix model lead to introduce an interpolated continuous version of the 2-Toda hierarchy: the *dispersionful* 2-Toda hierarchy (see for instance [4]). On the other hand, the leading contribution to the large N-limit (planar contribution) motivated the introduction of a *classical* version of the Toda hierarchy [4] which is known as the *dispersionless* 2-Toda (d2-Toda) hierarchy.

Laplacian growth processes describe evolutions of two-dimensional domains driven by harmonic fields. It was shown in [5] that the d2-Toda is a relevant integrable structure in Laplacian growth problems and conformal maps dynamics. For example, if a given analytic curve $\gamma\,(z = z(p),\, |p| = 1)$ is the boundary of a simply-connected bounded domain, then γ evolves with respect to its harmonic moments according to a solution of the d2-Toda hierarchy. These solutions are characterized by the string equations

$$\bar{z} = m, \quad \overline{m} = -z. \tag{1.3}$$

Here (z,m) and $(\bar{z},\bar{m})$ denote the two pairs of Lax-Orlov operators of the d2-Toda hierarchy. As it was noticed in [5]-[9], this integrable structure also emerges in the planar limit of the normal matrix model (1.2) and describes the evolution of the support of eigenvalues under a change of the parameters t_k of the potential.

The present paper is motivated by the recent discovery [10] of an integrable structure provided by the dispersionless AKNS hierarchy which describes the bubble break-off in Hele-Shaw flows. In this work we prove that this integrable structure is also characterized by the solution of a pair of string equations

$$z=\bar{z}, \quad m=\bar{m}, \tag{1.4}$$

of the d2-Toda hierarchy. Since the system (1.4) describes the planar limit of (1.1), it constitutes a common integrable structure arising in the Hermitian matrix model and the theory of Hele-Shaw flows.

Our strategy is inspired by previous results [11]-[12] on solution methods for dispersionless string equations. We also develop some useful standard technology of the theory of Lax equations in the context of the d2-Toda hierarchy.

The paper is organized as follows:

In the next section the basic theory of the d2-Toda hierarchy, the method of string equations and the solution of (1.4) are discussed. In Section 3 we show how the solution of (1.4) appears in the planar limit of the Hermitian matrix model and the Hele-Shaw bubble break-off processes studied in [10].

2 The dispersionless Toda hierarchy

2.1 String equations in the d2-Toda hierarchy

The dispersionless d2-Toda hierarchy [4] can be formulated in terms of two pairs (z,m) and $(\bar{z},\bar{m})$ of Lax-Orlov functions, where z and $\bar{z}$ are series in a complex variable p of the form

$$z=p+u+\frac{u_1}{p}+\cdots, \quad \bar{z}=\frac{v}{p}+v_0+v_1 p+\cdots, \tag{2.1}$$

while m and $\bar{m}$ are series in z and $\bar{z}$ of the form

$$m:=\sum_{j=1}^{\infty} j t_j z^{j-1}+\frac{x}{z}+\sum_{j\geqslant 1}\frac{S_{j+1}}{z^{j+1}}, \quad \bar{m}:=\sum_{j=1}^{\infty} j\bar{t}_j \bar{z}^{j-1}-\frac{x}{\bar{z}}+\sum_{j\geqslant 1}\frac{\bar{S}_{j+1}}{\bar{z}^{j+1}}. \tag{2.2}$$

The coefficients in the expansions (2.1) and (2.2) depend on a complex variable x and two infinite sets of complex variables $\mathrm{t}:=(t_1,t_2,\dots)$ and $\bar{\mathrm{t}}:=(\bar{t}_1,\bar{t}_2,\dots)$. The d2-Toda hierarchy is encoded in the equation

$$\mathrm{d}z\wedge\mathrm{d}m=\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{m}=\mathrm{d}\Big(\log p\,\mathrm{d}x+\sum_{j=1}^{\infty}\Big((z^j)_+\,\mathrm{d}t_j+(\bar{z}^j)_-\,\mathrm{d}\bar{t}_j\Big)\Big). \tag{2.3}$$

Here the $(\pm)$ parts of p-series denote the truncations in the positive and strictly negative power terms, respectively. As a consequence there exist two *action* functions S and $\bar{S}$ verifying

$$\mathrm{d}S = m\,\mathrm{d}z + \log p\,\mathrm{d}x + \sum_{j=1}^{\infty}\Big((z^j)_+\,\mathrm{d}t_j + (\bar{z}^j)_-\,\mathrm{d}\bar{t}_j\Big),$$
$$\mathrm{d}\bar{S} = \bar{m}\,\mathrm{d}\bar{z} + \log p\,\mathrm{d}x + \sum_{j=1}^{\infty}\Big((z^j)_+\,\mathrm{d}t_j + (\bar{z}^j)_-\,\mathrm{d}\bar{t}_j\Big),$$

and such that they admit expansions

$$S = \sum_{j=1}^{\infty} t_j z^j + x\log z - \sum_{j\geqslant 1}\frac{S_{j+1}}{jz^j}, \quad \bar{S} = \sum_{j=1}^{\infty}\bar{t}_j\bar{z}^j - x\log\bar{z} - \bar{S}_0 + \sum_{j\geqslant 1}\frac{\bar{S}_{j+1}}{j\bar{z}^{j+1}}. \tag{2.4}$$

From (2.3) one derives the d2-Toda hierarchy in Lax form

$$\frac{\partial\mathscr{K}}{\partial t_j} = \{(z^j)_+,\mathscr{K}\}, \quad \frac{\partial\mathscr{K}}{\partial\bar{t}_j} = \{(\bar{z}^j)_-,\mathscr{K}\}, \tag{2.5}$$

where $\mathscr{K} = z, m, \bar{z}, \bar{m}$, and we are using the Poisson bracket $\{f,g\} := p\,(f_p\,g_x - f_x\,g_p)$.

The following result was proved by Takasaki and Takebe (see [4]):

Theorem 1. *Let* $(P(z,m),Q(z,m))$ *and* $(\bar{P}(\bar{z},\bar{m}),\bar{Q}(\bar{z},\bar{m}))$ *be functions such that*

$$\{P,Q\} = \{z,m\}, \quad \{\bar{P},\bar{Q}\} = \{\bar{z},\bar{m}\}.$$

If (z,m) *and* $(\bar{z},\bar{m})$ *are functions which can be expanded in the form* (2.1)-(2.2) *and satisfy the pair of constraints*

$$P(z,m) = \bar{P}(\bar{z},\bar{m}), \quad Q(z,m) = \bar{Q}(\bar{z},\bar{m}), \tag{2.6}$$

then they verify $\{z,m\} = \{\bar{z},\bar{m}\} = 1$ *and are solutions of the Lax equations* (2.5) *for the d2-Toda hierarchy .*

Constraints of the form (2.6) are called *dispersionless string equations*. In this paper we are concerned with the system (1.4). The first equation $z = \bar{z}$ of (1.4) defines the 1-Toda reduction of the d2-Toda hierarchy

$$z = \bar{z} = p + u + \frac{v}{p}, \tag{2.7}$$

where

$$u = \partial_x S_2, \quad \log v = -\partial_x\bar{S}_0. \tag{2.8}$$

As a consequence the Lax equations (2.5) imply that u and v depend on $(\mathrm{t},\bar{\mathrm{t}})$ through the combination $\mathrm{t}-\bar{\mathrm{t}}$.

Due to (2.7) there are two branches of p as a function of z

$$p(z) = \frac{1}{2}\Big((z-u) + \sqrt{(z-u)^2-4v}\Big) = z - u - \frac{v}{z} + \cdots$$
$$\bar{p}(z) = \frac{1}{2}\Big((z-u) - \sqrt{(z-u)^2-4v}\Big) = \frac{v}{z} + \cdots. \tag{2.9}$$

To characterize the members of the d1-Toda hierarchy of integrable systems as well as to solve the string equations (1.4) it is required to determine $(z^j)_-(p(z))$ and $(z^j)_+(\bar{p}(z))$ in terms of (u,v). By using (2.7) it is clear that there are functions $(\alpha_j, \beta_j, \bar{\alpha}_j, \beta_j)$, which depend polynomially, in z such that

$$\partial_{t_j} S(z) = (z^j)_+(p(z)) = \alpha_j + \beta_j\, p(z), \quad \partial_{\bar{t}_j} S(z) = (z^j)_-(p(z)) = \bar{\alpha}_j + \bar{\beta}_j\, p(z),$$
$$\partial_{t_j} \bar{S}(z) = (z^j)_+(\bar{p}(z)) = \alpha_j + \beta_j\, \bar{p}(z), \quad \partial_{\bar{t}_j} \bar{S}(z) = (z^j)_-(\bar{p}(z)) = \bar{\alpha}_j + \bar{\beta}_j\, \bar{p}(z),$$

and

$$\bar{\alpha}_j = z^j - \alpha_j, \quad \bar{\beta}_j = -\beta_j. \tag{2.10}$$

Now we have

$$\alpha_j + \beta_j\, p(z) = \partial_{t_j} S(z) = z^j + \mathscr{O}\Big(\frac{1}{z}\Big), \quad \alpha_j + \beta_j\, \bar{p}(z) = \partial_{t_j} \bar{S}(z) = -\partial_{t_j} \bar{S}_0 + \mathscr{O}\Big(\frac{1}{z}\Big), \tag{2.11}$$

so that

$$\alpha_j = \frac{1}{2}\Big(z^j - \partial_{t_j} \bar{S}_0 - (p+\bar{p})\,\beta_j\Big)_{\oplus}, \quad \beta_j = \Big(\frac{z^j}{p-\bar{p}}\Big)_{\oplus}, \tag{2.12}$$

where $(\ )_{\oplus}$ and $(\ )_{\ominus}$ stand for the projection of z-series on the positive and strictly negative powers, respectively. Thus, by introducing the generating function

$$R := \frac{z}{p-\bar{p}} = \frac{z}{\sqrt{(z-u)^2 - 4v}} = \sum_{k\geqslant 0} \frac{r_k(u,v)}{z^k}, \quad r_0 = 1. \tag{2.13}$$

we deduce

$$\begin{aligned}(z^j)_+(p(z)) &= z^j - \frac{1}{2}\,\partial_{t_j} \bar{S}_0 - \frac{z}{2R}\Big(z^{j-1} R\Big)_{\ominus} \\ &= z^j - \frac{1}{2}(\partial_{t_j} \bar{S}_0 + r_j) - \frac{1}{2z}(r_{j+1} - u r_j) + \mathscr{O}\Big(\frac{1}{z^2}\Big).\end{aligned} \tag{2.14}$$

Hence

$$\partial_{t_j} \bar{S}_0 = -r_j, \quad \partial_{t_j} S_2 = \frac{1}{2}(r_{j+1} - u r_j),$$

so that the equations of the $d1$-Toda hierarchy are given by

$$\partial_{t_j} u = \frac{1}{2}\,\partial_x (r_{j+1} - u r_j), \quad \partial_{t_j} v = v\,\partial_x r_j. \tag{2.15}$$

Furthermore, we have found

$$(z^j)_-(p(z)) = -\frac{1}{2} r_j + \frac{z}{2R}\Big(z^{j-1} R\Big)_{\ominus}, \quad (z^j)_+(\bar{p}(z)) = r_j + (z^j)_-(p(z)). \tag{2.16}$$

Hence, the first terms of their asymptotic expansions as $z \to \infty$ are

$$(z^j)_-(p(z)) = \frac{1}{2z}(r_{j+1} - u r_j) + \mathscr{O}\Big(\frac{1}{z^2}\Big), \quad (z^j)_+(\bar{p}(z)) = r_j + \frac{1}{2z}(r_{j+1} - u r_j) + \mathscr{O}\Big(\frac{1}{z^2}\Big). \tag{2.17}$$

Notice that since $r_0 = 1$ and $r_1 = u$, these last equations hold for $j \geqslant 0$.

2.2 Hodograph solutions of the 1-dToda hierarchy

In the above paragraph we have used the first string equation of (1.4). Let us now deal with the second one. To this end we set

$$m = \overline{m} = \sum_{j=1}^{\infty} j\, t_j\, (z^{j-1})_+ + \sum_{j=1}^{\infty} j\, \overline{t_j}\, (z^{j-1})_-,$$

which leads to the following expressions for the Orlov functions $(m, \overline{m})$

$$\begin{aligned} m(z) &= \sum_{j=1}^{\infty} j\, t_j\, z^{j-1} + \sum_{j=1}^{\infty} j\, (\overline{t_j} - t_j)\, (z^{j-1})_-(p(z)), \\ \overline{m}(z) &= \sum_{j=1}^{\infty} j\, \overline{t_j}\, z^{j-1} - \sum_{j=1}^{\infty} j\, (\overline{t_j} - t_j)\, (z^{j-1})_+(\overline{p}(z)). \end{aligned} \tag{2.18}$$

In order to apply Theorem 1 we have to determine u and v and ensure that $(m, \overline{m})$ verify the correct asymptotic form (2.1)-(2.2). Both things can be achieved by reducing (2.18) to the form

$$\begin{aligned} \frac{x}{z} + \sum_{j\geqslant 2} \frac{1}{z^j} S_j &= \sum_{j=1}^{\infty} j\, (\overline{t_j} - t_j)\, (z^{j-1})_-(p(z)), \\ -\frac{x}{z} + \sum_{j\geqslant 2} \frac{1}{z^j} \overline{S}_j &= -\sum_{j=1}^{\infty} j\, (\overline{t_j} - t_j)\, (z^{j-1})_+(\overline{p}(z)), \end{aligned} \tag{2.19}$$

and equating coefficients of powers of z. Indeed, from (2.17) we see that identifying the coefficients of z^{-1} in both sides of the two equations of (2.19) yields the same relation. This equation together with the one supplied by identifying the coefficients of the constant terms in the second equation of (2.19) provides the following system of *hodograph-type* equations to determine (u, v)

$$\begin{cases} \sum_{j=1}^{\infty} j\, (\overline{t_j} - t_j) r_{j-1} = 0, \\ \dfrac{1}{2} \sum_{j=1}^{\infty} j\, (\overline{t_j} - t_j)\, r_j = x. \end{cases} \tag{2.20}$$

It can be rewritten as

$$\begin{cases} \displaystyle\oint_\gamma \frac{dz}{2\pi i} \frac{V_z}{\sqrt{(z-u)^2 - 4v}} = 0, \\ \displaystyle\oint_\gamma \frac{dz}{2\pi i} \frac{z V_z}{\sqrt{(z-u)^2 - 4v}} = -2x, \end{cases} \tag{2.21}$$

where γ is a large enough positively oriented closed path and V_z denotes the derivative with respect to z of the function

$$V(z, \mathrm{t} - \overline{\mathrm{t}}) := \sum_{j=1}^{\infty} (t_j - \overline{t_j})\, z^j. \tag{2.22}$$

The remaining equations arising from (2.19) characterize the functions $S_j^{(0)}$ and $\overline{S}_j^{(0)}$ for $j \geqslant 1$ in terms of (u,v). Therefore we have characterized a solution (z,m) and $(\overline{z},\overline{m})$ of the system of string equations (1.4) verifying the conditions of Theorem 1 and, consequently, it solves the d1-Toda hierarchy.

3 Planar limit of the Hermitian matrix model and bubble break-off in Hele-Shaw flows

3.1 The Hermitian matrix model

If we write the partition function (1.1) of the Hermitian matrix model in terms of eigenvalues and slow variables $\mathrm{t} := \varepsilon t$, where $\varepsilon = 1/N$, we get

$$Z_n(N\mathrm{t}) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} \Big(d\,x_k\, e^{NV(x_k,\mathrm{t})} \Big) (\Delta(x_1,\cdots,x_n))^2, \quad V(z,\mathrm{t}) := \sum_{k\geqslant 1} t_k z^k. \tag{3.1}$$

The large N-limit of the model is determined by the asymptotic expansion of $Z_n(N\mathrm{t})$ for $n = N$ as $N \to \infty$

$$Z_N(N\mathrm{t}) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \Big(d\,x_k\, e^{NV(x_k,\mathrm{t})} \Big) (\Delta(x_1,\cdots,x_N))^2, \tag{3.2}$$

It is well-known [3] that $Z_n(t)$ is a τ-function of the semi-infinite 1-Toda hierarchy , then there exists a τ-function $\tau(\varepsilon,x,\mathrm{t})$ of the dispersionful 1-Toda hierarchy verifying

$$\tau(\varepsilon,\varepsilon n,\mathrm{t}) = Z_n(N\mathrm{t}), \tag{3.3}$$

and consequently

$$\tau(\varepsilon,1,\mathrm{t}) = Z_N(N\mathrm{t}). \tag{3.4}$$

Hence the large N-limit expansion of the partition function

$$\mathbb{Z}_N(N\mathrm{t}) = \exp\Big(N^2\,\mathbb{F}\Big), \quad \mathbb{F} = \sum_{k\geqslant 0} \frac{1}{N^{2k}} F^{(2k)}, \tag{3.5}$$

is determined by a solution of the dispersionful 1-Toda hierarchy at $x = 1$.

As a consequence of the above analysis one concludes that the leading term (planar limit) $F^{(0)}$ is determined by a solution of the 1-dToda hierarchy at $x = 1$. Furthermore, the leading terms of the N-expansions of the main objects of the hermitian matrix model can be expressed in terms of quantities of the 1-dToda hierarchy. For example, in the *one-cut* case , the density of eigenvalues

$$\rho(z) = M(z)\sqrt{(z-a)(z-b)},$$

is supported on a single interval $[a,b]$. These objects are related to the leading term $W^{(0)}$ of the one-point correlator [13]

$$W(z) := \frac{1}{N} \sum_{j\geqslant 0} \frac{1}{z^{j+1}} \langle trM^j \rangle = \frac{1}{z} + \frac{1}{N^2} \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \log Z_N(N\mathrm{t})}{\partial t_j},$$

in the form

$$W^{(0)} = -\frac{1}{2}V_z(z) + i\pi\rho(z).$$

On the other hand, it can be proved (see for instance [14]) that

$$W^{(0)} = m(z,1,\mathrm{t}) - \sum_{j=1}^{\infty} j t_j z^{j-1}, \tag{3.6}$$

so that (2.16) and (2.18) yield

$$\begin{aligned} -\frac{1}{2}V_z(z) + i\pi\rho(z) &= -\sum_{j=1}^{\infty} j t_j (z^{j-1})_-(p(z)) \\ &= \frac{1}{2}\sum_{j=1}^{\infty} j t_j r_{j-1} - \frac{1}{2}\sum_{j=1}^{\infty} j t_j z^{j-1} + \frac{1}{2}(p-\overline{p})\sum_{j=2}^{\infty} j t_j \left(z^{j-2}R\right)_{\oplus}, \end{aligned} \tag{3.7}$$

Since we are setting $\overline{t_j} = 0, \forall j \geqslant 1$, according to the first hodograph equation (2.20) the first term in the last equation vanishes. Therefore the density of eigenvalues and its support $[a,b]$ are characterized by

$$\begin{aligned} &\rho(z) := \frac{1}{2\pi i}\left(\frac{V_z}{\sqrt{(z-a)(z-b)}}\right)_{\oplus}\sqrt{(z-a)(z-b)}, \\ &a := u - 2\sqrt{v}, \quad b := u + 2\sqrt{v}, \end{aligned} \tag{3.8}$$

where we set $x = 1$ in all the x-dependent functions. Observe that according to (3.7)

$$i\pi\rho(z) = \frac{1}{2}V_z(z) + \frac{x}{z} + \mathscr{O}\left(\frac{1}{z^2}\right), \quad z \to \infty, \tag{3.9}$$

so that the constraint $x = 1$ means that the density of eigenvalues is normalized on its support

$$\int_a^b \rho(z)\,dz = 1.$$

Moreover, from (2.21) we obtain

$$\oint_\gamma \frac{dz}{2\pi i}\frac{V_z}{\sqrt{(z-a)(z-b)}} = 0, \quad \oint_\gamma \frac{dz}{2\pi i}\frac{zV_z}{\sqrt{(z-a)(z-b)}} = -2, \tag{3.10}$$

with γ being a positively oriented closed path encircling the interval $[a,b]$. These are the quations which determine the zero-genus contribution or planar limit to the partition function of the hermitian model [14]-[18].

3.2 Bubble break-off in Hele-Shaw flows

A Hele-Shaw cell is a narrow gap between two plates filled with two fluids: say oil surrounding one or several bubbles of air. Let D denote the domain in the complex plane $\mathbb{C}$ of the variable λ occupied by the air bubbles. By assuming that D is an *algebraic domain* [10], the boundary γ of D is characterized by a *Schwarz function* $\mathbb{S} = \mathbb{S}(\lambda)$ such that

$$\lambda^* = \mathbb{S}(\lambda), \quad \lambda \in \gamma. \tag{3.11}$$

The geometry of the domain $\mathbb{C} - D$ is completely encoded in $\mathbb{S}$ and it can be conveniently described in terms of the *Schottky double* [9]: a Riemann surface $\mathscr{R}$ resulting from gluing two copies $H_\pm$ of $\mathbb{C} - D$ trough γ, adding two points at infinity $(\infty, \overline{\infty})$ and defining the complex coordinates

$$\begin{cases} \lambda_+(\lambda) = \lambda, & \lambda \in H_+, \\ \lambda_-(\lambda) = \lambda^*, & \lambda \in H_-. \end{cases}$$

In particular $\mathbb{S}\, d\lambda$ can be extended to a unique meromorphic differential ω on $\mathscr{R}$.

The evolution of γ is governed by D'Arcy law: the velocity in the oil domain is proportional to the gradient of the pressure. In the absence of surface tension, pressure is continuous across γ and then if the bubbles are assumed to be kept at zero pressure, we are lead to the Dirichlet boundary problem

$$\begin{cases} \triangle \mathscr{P} = 0, & \text{on } \mathbb{C} - D, \\ \mathscr{P} = 0 & \text{on } \gamma, \\ \mathscr{P} \to -\log|z|, & z \to \infty. \end{cases} \tag{3.12}$$

If one assumes D'Arcy law in the form $\vec{v} = -2\vec{\nabla}\mathscr{P}$, then by introducing the function

$$\Phi(\lambda) := \xi(\lambda) + i\,\mathscr{P}(\lambda), \tag{3.13}$$

where ξ and $\mathscr{P}$ are the *stream function* and the pressure, respectively, D'Arcy law can be rewritten as

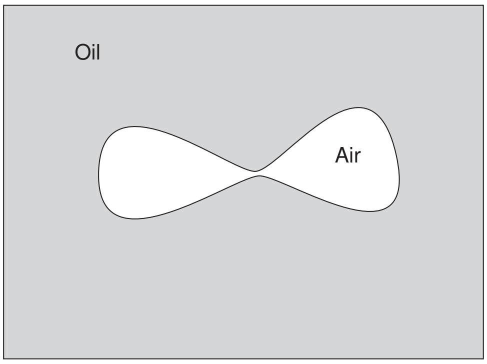
$$\partial_t \mathbb{S} = 2\, i\, \partial_\lambda \Phi, \tag{3.14}$$

where t stands for the time variable.

In the set-up considered in [10] air is drawn out from two fixed points of a simply-connected air bubble making the bubble breaks into two emergent bubbles with highly curved tips. Before the break-off the interface oil-air remains free of cusp-like singularities and develops a smooth neck. As it is shown in [9]-[10], the condition for bubbles to be at equal pressure implies that the integral

$$\Pi := \frac{1}{2} \oint_\beta \omega,$$

where ω is the meromorphic extension of $\mathbb{S}\, d\lambda$ to $\mathscr{R}$ and β is a cycle connecting the bubbles, is a constant of the motion. Since at break-off β contracts to a point, it is obvious that a necessary condition for break-off is that Π vanishes.



The following pair of complex-valued functions were introduced in [10] to describe the bubble break-off near the breaking point

$$X(\lambda) := \frac{1}{2}\Big(\lambda + \mathbb{S}(\lambda)\Big), \quad Y(\lambda) := \frac{1}{2i}\Big(\lambda - \mathbb{S}(\lambda)\Big). \tag{3.15}$$

They analytically extend the Cartesian coordinates (X,Y) of the interface γ

$$X = Re\,\lambda, \quad Y = Im\,\lambda, \quad \lambda \in \gamma, \tag{3.16}$$

and allow to write the evolution law (3.14) in the form

$$\partial_t Y(X) = -\partial_X \Phi(X). \tag{3.17}$$

The analysis of [10] concludes that after the break-off the local structure of a small part of the interface containing the tips of the bubbles falls into universal classes characterized by two even integers $(4n,2)$, $n \geqslant 1$, and a finite number $2n$ of real deformation parameters t_k. By assuming symmetry of the curve with respect to the X-axis, the general solution for the curve and the potential in the $(4n,2)$ class are

$$Y := \left(\frac{U_X}{\sqrt{(X-a)(X-b)}}\right)_{\oplus} \sqrt{(X-a)(Y-b)}, \quad \Phi = -\sqrt{(X-a)(Y-b)}, \tag{3.18}$$

where a and b are the positions of the bubbles tips and

$$U(X,\mathrm{t}) := \sum_{j=1}^{2n} t_{j+1} X^{j+1}. \tag{3.19}$$

Here the subscript $\oplus$ denotes the projection of X-series on the positive powers. Due to the physical assumptions of the problem, the function Y inherates two conditions for its expansion as $X \to \infty$

$$Y(X) = \sum_{j=1}^{2n} (j+1) t_{j+1} X^j + \sum_{j=0}^{\infty} \frac{Y_n}{X^n}. \tag{3.20}$$

which determine the positions a, b of the tips. The conditions are

1. From (3.12) $\Phi \to -i \log \lambda$ as $\lambda \to \infty$. Hence (3.17) implies that the constant term Y_0 in (3.20) should be equal to t.

2. The coefficient Y_1 in front of X^{-1} turns to be equal to Π, so that it must vanish for a break-off [10].

As it was shown in [10], imposing these two conditions on (3.20) leads to a pair of hodograph equations which arise in the dispersionless AKNS hierachy. However, from (3.18) it is straightforward to see that these equations coincide with the hodograph equations (2.20) associated with the system of string equations (1.4) provided one sets

$$\begin{gathered} X = z, \quad Y = 2m - V_z, \quad \Phi = z - u - 2p, \\ t_j = 0, \quad \forall j \geqslant 2n+2; \quad t = t_1, \quad x = \frac{\Pi}{2} = 0. \end{gathered} \tag{3.21}$$

For instance, we observe that the evolution law (3.17) derives in a very natural form from the d1-Toda hierarchy. Indeed, from (2.9) and (3.18) we have

$$p = \frac{1}{2}(z - u - \Phi),$$

so that (3.21) implies

$$\partial_t Y = 2\partial_{t_1} m(z) - 1 = 2\partial_z (z)_+ - 1 = 2\partial_z (p+u) - 1 = -\partial_z \Phi = -\partial_X \Phi.$$

In this way the integrable structure associated to the system of string equations (1.4) of the d2-Toda hierarchy manifests a duality between the planar limit of the Hermitian matrix model and the bubble break-off in Hele-Shaw cells. According to this relationship the density of eigenvalues ρ and the end-points a, b of its support in the Hermitian model are identified with the interface function Y and the positions of the bubbles tips, respectively, in the Hele-Shaw model.

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