Symmetry Analysis and Solutions for a Generalization of a Family of BBM Equations

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Abstract

In this paper, the family of BBM equation with strong nonlinear dispersive $B(m,n)$ is considered. We apply the classical Lie method of infinitesimals. The symmetry reductions are derived from the optimal system of subalgebras and lead to systems of ordinary differential equations. We obtain for special values of the parameters of this equation, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic function solutions and their degenerative solutions (soliton, kink and compactons).

1   Introduction

Benjamin *et al* [2] proposed the regularised long wave (RLW) equation, or Benjamin-Bona-Mahony equation (BBM),

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

as an alternative model to the Korteweg–de Vries equation for the long wave motion in nonlinear dispersive systems. These authors argued that both equations are valid at the same level of approximation, but that BBM does have some advantages over the KdV from the computational mathematics viewpoint.

In order to understand the role of nonlinear dispersion in the formation of patterns in an undular bore, Yalong [17] introduced and studied a family of BBM-like equations with nonlinear dispersion, $B(m,n)$ equations

$$u_t + (u^m)_x - (u^n)_{xxt} = 0, \qquad m,n > 1.$$

In [17], the exact solitary-wave solutions with compact support and exact special solutions with solitary patterns of the equations were derived.

In [12] the authors introduced the family of BBM equation with strong nonlinear dispersive B(m,n) equation:

$$u_t + u_x + a\,(u^m)_x + (u^n)_{xxt} = 0, \tag{1.1}$$

by using an algebraic method and they obtained solitary pattern solutions. The case $n = 1$ and $m = 2$ corresponds to the BBM equation, [2]. This equation is an alternative to the Kortewegde

Vries (KdV) equation, and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. The BBM equation is not only convenient for shallow water waves but also for hydromagnetic and acoustic waves, and therefore it has some advantages compared with the KdV equation.

Clarkson [6] showed that the similarity reduction of the equation (1.1) for $m = 3$, $n = 1$ and $a = \frac{1}{3}$, obtained by using the classical Lie group method reduces the partial differential equation (PDE) to an ordinary differential equation (ODE) of Painlevé type; whereas the partial differential equation doesn't possesses the Painlevé property for partial differential equations as defined by Weiss *et al* [13]. The author proved that the only non-constant similarity reductions of this equation obtainable either using the classical Lie method or the direct method, due to Clarkson and Kruskal [7], are the travelling wave solutions.

In this paper we study similarity reductions of the equation

$$u_t + bu_x + a\,(u^m)_x + (u^n)_{xxt} = 0, \tag{1.2}$$

where a, b are constants, $b \neq 0$, and $n, m \in \mathbb{R}^*$ with m or $n \neq 1$, by using the Lie method of infinitesimals. The fundamental basis of this method is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. Though the method is entirely algorithmic, it involves a large amount of algebra and of auxiliary calculations. Some symbolic manipulations programs have been developed to simplify the calculations. We use the MACSYMA program symmgrp.max [5] and we have checked the results by using the MATH-EMATICA program SYM.nb [8, 9]. In order to find all invariant solutions with respect to *s*-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order *s*. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*. For PDEs with two independent variables a single group reduction transforms the PDE into a ODEs, which are generally easier to solve. The required theory and description of the method can be found in [3, 10, 11, 14].

The outline is as follows: in §2 we obtain the symmetry reductions, similarity variables and reduced ordinary differential equations (depending on a, b, m and n); in §3 we derive, for special values of the parameters, exact solutions which can be expressed by various single and combine nondegenerative Jacobi elliptic function solutions and their degenerative solutions (soliton, kink and compactons); finally, in §4 some conclusions are presented.

2 Lie Symmetries

To apply the classical method to Eq. (1.2) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \varepsilon \xi(x, t, u) + O(\varepsilon^2),$$
$$t^* = t + \varepsilon \tau(x, t, u) + O(\varepsilon^2),$$
$$u^* = u + \varepsilon \eta(x, t, u) + O(\varepsilon^2),$$

where ε is the group parameter. We require that this transformation leaves invariant the set of solutions of (1.2). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \tag{2.1}$$

Invariance of Eq. (1.2) under a Lie group of point transformations with infinitesimal generator (2.1) leads to a set of twenty six determining equations. Solving this system we obtain $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta = \dfrac{\alpha(x,t)}{u^{n-1}} - \dfrac{k_1 u}{2n} + \dfrac{\xi_x u}{2n}$ where ξ, τ and α are related by the following conditions:

$$\xi_{xxx} n^2 u^{2n} + k_1 n u^{n+1} + 3\xi_x n u^{n+1} - k_1 u^{n+1} + \xi_x u^{n+1} - 2\alpha n^2 u + 2\alpha n u = 0,$$
$$a\xi_{xx} m u^{m+n} + 2a\alpha_x n m u^m + b\xi_{xx} u^{n+1} + 2\alpha_{txx} n^2 u^n + 2\alpha_x b n u + 2\alpha_t n u = 0,$$
$$-ak_1 m^2 u^{m+n} + a\xi_x m^2 u^{m+n} + ak_1 n m u^{m+n} + 2a\tau_t n m u^{m+n} + a\xi_x n m u^{m+n}$$
$$+2a\alpha n m^2 u^m - 2a\alpha n^2 m u^m + bk_1 n u^{n+1} + 2b\tau_t n u^{n+1}$$
$$+b\xi_x n u^{n+1} - bk_1 u^{n+1} + b\xi_x u^{n+1} - 2\alpha b n^2 u + 2\alpha b n u = 0.$$

The solutions of this system depend on the parameters of equation (1.2) and we can distinguish the following cases:

1. If a and b are arbitrary constants, the only symmetries admitted by (1.2) are the group of space and time translations, which are defined by the infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t.$$

• For $\lambda \mathbf{v}_1 + \mathbf{v}_2$ the similarity variables and similarity solution are:

$$\begin{aligned} z &= x - \lambda t \\ u &= h(z) \end{aligned} \tag{2.2}$$

where $h(z)$ satisfies

$$\lambda (h^n)''' + \lambda h' - amh^{m-1}h' - bh' = 0.$$

This equation, after integrating once with respect to z, can be reduced to

$$\lambda (h^n)'' = ah^m + (b-\lambda)h + k_1, \tag{2.3}$$

where k_1 is an integrating constant.

2. The cases for which Eq.(1.2) with $b \neq 0$ have extra symmetries are given in the Table 1.

Table 1: Symmetries for a Generalization of a Family of BBM Equations.

i	constants	V_3^i	$\mathbf{v}_\infty^i$
1	$a \cdot (m-1) = 0$	$(n-1)x\partial_x + (n-1)t\partial_t + 2u\partial_u$	
2	$m = 1, a = -b$	$(n-1)x\partial_x + 2u\partial_u$	$\tau(t)\partial_t$
3	$m = 2, n = 1$	$-t\partial_t + \left(u + \dfrac{b}{2a}\right)\partial_u$	

where $\tau(t)$ is an arbitrary function.

In order to determine solutions of PDE (1.2) that are not equivalent by the action of the group, we must calculate the one-dimensional optimal system [10]. Next we construct a table showing the separate adjoint actions of each element as it acts over all the other elements. This construction is done by summing the Lie series.

Table 2: Commutator table for the Lie algebra $\{\mathbf{v}_i^1\}$.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^1$
$\mathbf{v}_1$	0	0	$(n-1)\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$(n-1)\mathbf{v}_2$
$\mathbf{v}_3^1$	$-(n-1)\mathbf{v}_1$	$-(n-1)\mathbf{v}_2$	0

Table 3: Adjoint table for the Lie algebra $\{\mathbf{v}_i^1\}$.

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^1$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^1 - \varepsilon(n-1)\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^1 - \varepsilon(n-1)\mathbf{v}_2$
$\mathbf{v}_3^1$	$e^{(n-1)\varepsilon}\mathbf{v}_1$	$e^{(n-1)\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3^1$

Table 4: Commutator table for the Lie algebra $\{\mathbf{v}_i^2\}$.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^2$
$\mathbf{v}_1$	0	0	$(n-1)\mathbf{v}_1$
$\mathbf{v}_2$	0	0	0
$\mathbf{v}_3^2$	$-(n-1)\mathbf{v}_1$	0	0

Table 5: Adjoint table for the Lie algebra $\{\mathbf{v}_i^2\}$.

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^2$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^2 - (n-1)\varepsilon\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^2$
$\mathbf{v}_3^2$	$e^{(n-1)\varepsilon}\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^2$

Table 6: Commutator table for the Lie algebra $\{\mathbf{v}_i^3\}$.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$-\mathbf{v}_2$
$\mathbf{v}_3^3$	0	$\mathbf{v}_2$	0

Table 7: Adjoint table for the Lie algebra $\{\mathbf{v}_i^3\}$.

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3^3 + \varepsilon\mathbf{v}_2$
$\mathbf{v}_3^3$	$\mathbf{v}_1$	$e^{-\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3^3$

The generators of the nontrivial one-dimensional optimal system are the set

$$\mathbf{v}_1, \quad \lambda\mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_3^1, \quad \lambda\mathbf{v}_2 + \mathbf{v}_3^2, \quad \lambda\mathbf{v}_1 + \mathbf{v}_3^3.$$

Since equation (1.2) has additional symmetries and the reductions that correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ have already been derived, we must only determine the similarity variables and similarity solutions corresponding to the remaining generators:

• For $\mathbf{v}_3^1$ the similarity variables and similarity solution are:

$$z = \frac{x}{t}, \qquad u = h(z) t^{\frac{2}{n-1}},$$

the reduced ODE is

$$(n-1)\, z\, (h^n)''' - 2\, (h^n)'' + h'\, (n-1)\, (z - a - b) - 2h = 0.$$

For $n = 2$ and $b = -a$, after integrating once with respect to z, $h(z)$ must satisfy

$$h h'' - 2\, (h')^2 - h^4 + k_1 z h^3 = 0, \tag{2.4}$$

where k_1 is an arbitrary constant.

• For $\lambda \mathbf{v}_2 + \mathbf{v}_3^2 + \mathbf{v}_\infty^2$ the similarity variables and similarity solution are:

$$z = \delta(t) - lnx, \qquad u = h(z)\, e^{\frac{2}{n-1}},$$

where $\delta = \int \frac{n-1}{\lambda + \tau(t)} dt$ and $h(z)$ satisfies the ODE

$$
\begin{aligned}
&\left(h^{n+2} n^3 - 2 h^{n+2} n^2 + h^{n+2} n\right) h''' + \left(3 h^{n+1} h' n^4 + h^n \left(-9 h h' - 3 h^2\right) n^3 + h^n \left(9 h h'\right.\right. \\
&\left.+ 2 h^2\right) n^2 + h^n \left(h^2 - 3 h h'\right) n) h'' + h^n (h')^3 n^5 + h^n \left(-5 (h')^3 - 3 h (h')^2\right) n^4 \\
&+ h^n \left(9 (h')^3 + 5 h (h')^2 + 2 h^2 h'\right) n^3 + \left(h^n \left(-7 (h')^3 - h (h')^2 + 2 h^2 h'\right) + h^3 h'\right) n^2 \\
&+ \left(h^n \left(2 (h')^3 - h (h')^2\right) - 2 h^3 h'\right) n + h^3 h' = 0.
\end{aligned}
$$

• For $\lambda \mathbf{v}_1 + \mathbf{v}_3^3$ the similarity variables and similarity solution are:

$$z = x + \lambda \ln(t), \qquad u = \frac{h(z)}{t} - \frac{b}{2a},$$

and $h(z)$ satisfies the ODE

$$\lambda h''' - h'' + \lambda h' + 2 a h h' - h = 0. \tag{2.5}$$

Equation (2.5) is invariant under translations, this allow us to reduce the order by one. By means of the change of variables $\{y = h, g = \frac{1}{h'}\}$, the following second order ordinary differential equation for $g(y)$ is obtained:

$$\lambda g g'' - \lambda g^4 + k_1 g^5 - g^2 g' - 3\lambda (g')^2 - 2 a y g^4 + y g^5 = 0.$$

3 Exact Solutions

By making the change of variables

$$h^n = y \tag{3.1}$$

equation (2.3) becomes

$$\lambda y'' = a y^{\frac{m}{n}} + (b - \lambda) y^{\frac{1}{n}} + k_1. \tag{3.2}$$

After multiplying (3.2) by $2y'$ and integrating once with respect to z we get

$$\lambda (y')^2 = \frac{2an}{m+n} y^{\frac{m}{n}+1} + \frac{2(b-\lambda)n}{n+1} y^{\frac{1}{n}+1} + 2k_1 y + k_2, \tag{3.3}$$

where k_2 is an integrating constant.

Let us assume that equation (3.3) has solution of the form

$$y(z) = \alpha f^\beta(z), \tag{3.4}$$

where α and β are parameters to be determined later.

By substituting (3.4) into (3.3) we obtain

$$(f')^2 = \frac{2an}{(m+n)\lambda\alpha\beta^2} f^{\frac{\beta m}{n}-\beta+2} + \frac{2(b-\lambda)n}{(n+1)\lambda\alpha\beta^2} f^{\frac{\beta}{n}-\beta+2} + \frac{2k_1}{\alpha\beta^2\lambda} f^{-\beta+2} + \frac{k_2}{\alpha^2\beta^2\lambda} f^{-2\beta+2}. \quad (3.5)$$

In the following we will determine the exponents and coefficients of equations (3.5). So that equation (3.5) is solvable in terms of Jacobi elliptic function, that is equation (3.5) becomes

$$(f')^2 = r + pf^2 + qf^4, \quad (3.6)$$

where r, p and q are constants.

Comparing the exponents and the coefficients of equations (3.5) and (3.6) we distinguish the following cases:

Case 1: If $k_1 = 0$ and $k_2 = 0$.

Subcase 1.1: $\beta = \dfrac{2}{m-1}$, $n = 1$ and $m \neq 1$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{4p\lambda}, \qquad a = \frac{(m+1)q(b-\lambda)}{2p}.$$

Subcase 1.2: $\beta = \dfrac{2m}{1-m}$ and $n = m$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{2m(m+1)q\lambda}, \qquad a = \frac{2mp(b-\lambda)}{(m+1)q}.$$

Subcase 1.3: $\beta = \dfrac{2n}{n-1}$ and $n = m$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{2m(m+1)r\lambda}, \qquad a = \frac{2mp(b-\lambda)}{(m+1)r}.$$

Subcase 1.4: $\beta = \dfrac{2n}{n-1}$ and $m = 2n-1$.

$$\alpha = \frac{(b-\lambda)(n-1)^2}{2n(n+1)r\lambda}, \qquad a = \frac{(3n-1)q(b-\lambda)}{(n+1)r}.$$

Subcase 1.5: $\beta = \dfrac{2}{1-m}$ and $n = 1$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{4p\lambda}, \qquad a = \frac{(m+1)r(b-\lambda)}{2p}.$$

Subcase 1.6: $\beta = \dfrac{2n}{n-1}$ and $m = 2n-1$.

$$\alpha = \frac{(n-1)^2(b-\lambda)}{2n(n+1)p\lambda}, \quad a = \frac{(3n-1)q(b-\lambda)}{(n+1)p\lambda}.$$

Subcase 1.7: $\lambda = b$, β is arbitrary and $m = n$.

$$\alpha \quad \text{is arbitrary}, \qquad a = \alpha\beta^2 pb.$$

Subcase 1.8: $\lambda = b$, $\beta = \frac{2n}{m-n}$.

$$\alpha = \frac{a(n-m)^2}{2bn(n+m)q}.$$

Subcase 1.9: $\lambda = b$, $\beta = \frac{2n}{n-m}$.

$$\alpha = \frac{a(n-m)^2}{2bn(n+m)r}.$$

Case 2: If $k_1 \neq 0$ and $k_2 = 0$.
Subcase 2.1: $\beta = 2$, $n = 1$ and $m = 2$.

$$\alpha = \frac{k_1}{2r\lambda}, \quad b = \frac{2k_1 p}{r} + \lambda, \quad a = \frac{3qk_1}{r}.$$

Subcase 2.2: $\beta = -2$, $n = 1$ and $m = 2$.

$$\alpha = \frac{k_1}{2q\lambda}, \quad b = \frac{2k_1 p}{q} + \lambda, \quad a = \frac{3k_1 r}{q}.$$

Subcase 2.3: $\beta = -2$, $n = m = \frac{1}{2}$.

$$\alpha = \frac{k_1}{2q\lambda}, \quad b = \frac{3k_1 r}{q} + \lambda, \quad a = \frac{2k_1 p}{q}.$$

Subcase 2.4: $\beta = 2$, $n = m = \frac{1}{2}$.

$$\alpha = \frac{k_1}{2r\lambda}, \quad b = \frac{3k_1 q}{r} + \lambda, \quad a = \frac{2k_1 p}{r}.$$

Case 3: If $k_1 = 0$ and $k_2 \neq 0$.
Subcase 3.1: $\beta = 1$, $n = 1$ and $m = 3$.

$$\alpha = \pm \left(\frac{k_2}{r\lambda} \right)^{\frac{1}{2}}, \quad b = \lambda \left[1 \pm p \left(\frac{k_2}{r\lambda} \right)^{\frac{1}{2}} \right], \quad a = \pm 2q\lambda \left(\frac{k_2}{r\lambda} \right)^{\frac{1}{2}}.$$

Subcase 3.2: $\beta = -1$, $n = 1$ and $m = 3$.

$$\alpha = \pm \left(\frac{k_2}{q\lambda} \right)^{\frac{1}{2}}, \quad b = \lambda \left[1 \pm p \left(\frac{k_2}{q\lambda} \right)^{\frac{1}{2}} \right], \quad a = \pm 2r\lambda \left(\frac{k_2}{q\lambda} \right)^{\frac{1}{2}}.$$

Subcase 3.3: $\beta = 1$, $n = m = \frac{1}{3}$.

$$\alpha = \pm \left(\frac{k_2}{r\lambda} \right)^{\frac{1}{2}}, \quad b = \lambda \left[1 \pm 2q \left(\frac{k_2}{r\lambda} \right)^{\frac{1}{2}} \right], \quad a = \pm p\lambda \left(\frac{k_2}{r\lambda} \right)^{\frac{1}{2}}.$$

Subcase 3.4: $\beta = -1$, $n = m = \frac{1}{3}$.

$$\alpha = \pm \left(\frac{k_2}{q\lambda} \right)^{\frac{1}{2}}, \quad b = \lambda \left[1 \pm 2r \left(\frac{k_2}{q\lambda} \right)^{\frac{1}{2}} \right], \quad a = \pm p\lambda \left(\frac{k_2}{q\lambda} \right)^{\frac{1}{2}}.$$

Since in all these cases, r, p and q are arbitrary constants, we may choose them properly such that the corresponding solution f of the ODE (3.6) are expressed in terms of the Jacobian elliptic functions. In the following we present some exact solutions.

• If $r = 1$, $p = -(1 + c^2)$, $q = c^2$, then

$$y = \alpha \left(\text{sn}(z|c) \right)^\beta,$$

where $\text{sn}(z|c)$ is the Jacobi elliptic function, is a solution of equation (3.3), [1].

From Subcase 3.1 for $\lambda = k_2$, $n = 1$, $m = 3$, $a = 2k_2 c^2$ and $b = -k_2 c^2$ we obtain the particular solution of equation (3.3)

$$y = \text{sn}(z|c).$$

From (3.1) and (2.2) for $c = 1$, $n = 1$, $m = 3$ and $a = -2b$ we obtain the exact solution of (1.2) given by

$$u(x,t) = \tanh(x + bt). \tag{3.7}$$

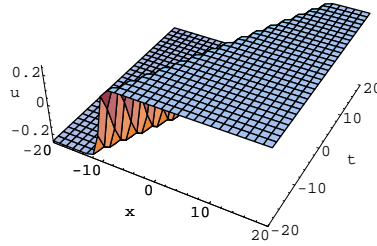If $b = -\frac{1}{2}$, (3.7) describes a kink solution (see Fig.1).

Figure 1: Solution (3.7) for $b = -\frac{1}{2}$.

From Subcase 2.4 for $\lambda = \frac{k_1}{2}$, $a = -2k_1(c^2 + 1)$ and $b = k_1(3c^2 + \frac{1}{2})$ we obtain the solution of equation (3.3)

$$y = \text{sn}^2(z|c).$$

From (3.1) and (2.2), for $c = 0$, $m = n = \frac{1}{2}$ and $a = -4b$, to yield

$$u(x,t) = \sin^4(x - bt). \tag{3.8}$$

• If $r = \frac{1-c^2}{4}$, $p = \frac{1+c^2}{2}$, $q = \frac{1-c^2}{4}$, $f = \text{nc}(z|c) \pm \text{sc}(z|c)$ is solution of equation (3.6), [1]. Then

$$y = \alpha \left[\text{nc}(z|c) \pm \text{sc}(z|c) \right]^\beta$$

is solution of equation (3.3), where α and β are arbitrary functions, $\text{nc}(z|c) = \frac{1}{\text{cn}(z|c)}$, $\text{sc}(z|c) = \frac{\text{sn}(z|c)}{\text{cn}(z|c)}$ where $\text{sn}(z|c)$ and $\text{cn}(z|c)$ are the first and the second Jacobian elliptic functions, respectively (the elliptic sine and the elliptic cosine).

From Subcase 1.7 for $\lambda = b$, $a = b\beta^2$ and $n = m$ we obtain the particular solution of equation (3.3)

$$y = [\text{nc}(z|1) \pm \text{sc}(z|1)]^\beta.$$

From (3.1) and (2.2) if $m = n$ and $a = b\beta^2$ we obtain the solution of equation (1.2)

$$u(x,t) = [\cosh(x - bt) \pm \sinh(x - bt)]^\beta. \tag{3.9}$$

• If $p = 1$ and $q = -1$,

$$y = \alpha\,(\text{cn}(z|1))^\beta$$

is solution of equation (3.3).

From subcase 1.1 for $\lambda = \dfrac{b(m-1)^2}{m^2 - 2m + 5}$, $n = 1$ and $a = -\dfrac{2b(m+1)}{m^2 - 2m + 5}$, the solution of equation (3.3) is

$$y = \text{sech}^{\frac{2}{m-1}}(z).$$

From (3.1) and (2.2) we obtain the solution of equation (1.2)

$$u(x,t) = \text{sech}^{\frac{2}{m-1}}(x - \lambda t). \tag{3.10}$$

For $m = 2$ and $\lambda = 1$, (3.10) describes a soliton moving along a line with constant velocity (see Fig.2).
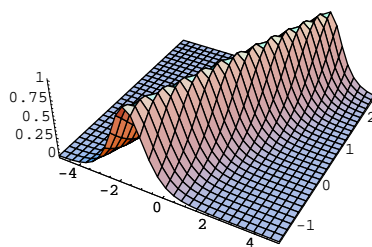

Figure 2: Solution (3.10) for $m = 2$, $\lambda = 1$ and $a = -6$.

Solutions (3.7) and (3.10) were first found in [12]. As far as we know, solutions (3.8) and (3.9) are new and have not been previously described in the literature.

4 Concluding remarks

In this paper we have seen a classification of symmetry reductions of a family of BBM equations, depending on the values of the constants a, b, n and m, by making use of the theory of symmetry reductions in differential equations. We have constructed all the invariant solutions with regard to the one-dimensional system of subalgebras. Besides the similarity reduction travelling wave solution, we find new similarity reductions for this family of equations. We have constructed all the ODE's to which (1.2) is reduced. We obtain for special values of the parameters of this equation, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic function solutions and their degenerative solutions (soliton, kink and compactons).

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