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Where Do Braid Statistics and Discrete Motion Meet Each Other?

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Abstract

We consider universal statistical properties of systems that are characterized by phase states with macroscopic degeneracy of the ground state. A possible topological order in such systems is described by non-linear discrete equations. We focus on the discrete equations which take place in the case of generalized exclusion principle statistics. We show that their exact solutions are quantum dimensions of the irreducible representations of certain quantum group. These solutions provide an example of the point where the generalized exclusion principle statistics and braid statistics meet each other. We propose a procedure to construct the quantum dimer models by means of projection of the knotted field configurations that involved braiding features of one-dimensional topology.

1 Introduction

The universal behavior of low-dimensional strongly correlated systems at low temperatures is determined to a great extent by the topology of the manifolds, determined by the ground state and the low-lying excitations. In strongly correlated electron liquids, with high degree of degeneracy of the ground state, such a manifold is presented by a collection of string-like structures, which may form arbitrarily tangled and knotted filaments. The processes of fusion and decay of the strings give rise to modifications of the tangle and, consequently, of the character of topological phase state order. Being a result of detailed study of electron liquids in the states with fractional Hall effect, this conclusion was supported recently by the results, obtained during studying dynamics of spin [46, 18, 36] and charge [39, 16] degrees of freedom in other low-dimensional electron systems.

It is well known that statistics of excitations in $(1+1)$- and $(2+1)$-dimensional systems is connected with the braid group [31, 49, 50]. Quantum states in such systems are classified by irreducible representations of the braid group, instead of the even (for bosons) or odd (for fermions) irreducible representations of the permutation group as (3+1)-dimensional systems. One-dimensional irreducible representations of the braid group correspond to Abelian anyon states, while multi-dimensional irreducible representations describe the non-Abelian states of anyons. Statistics of anyon excitations is called either braid statistics or fractional statistics, because it leads to the existence of particles with a fractional charge and spin. In the long-wavelength limit the description of the non-Abelian anyons is based on the effective action of topological field theories [53] containing the Chern-Simons term. An important particular is the case with gauge symmetry $SU(N)$ and integer quantized coupling constant k, or k-level Wess-Zumino-Witten-Novikov theory(see [9, 10]). This case corresponds to a find the representations of the quantum group $SU(N)_k$.

Another approach, based on the generalized exclusion principle [22], shows that the particle distribution functions can be derived from to the thermodynamic Bethe ansatz [54, 8, 38, 12, 20, 21, 55]. Determining the minima of the free energy of the model, one is lead to the Hirota's difference equation [52, 26]. From the theory of nonlinear equations, it is well known that this discrete equation yields to known integrable hierarchies of integrable nonlinear PDE's in the continuous limit [27]. In the derivation of discrete equation from the generalized exclusion principle, an important feature is the absence of any reference to the dimensionality of the space, unlike in braid statistics case. So, formally generalized exclusion principle statistics [22] may take place not only in low-dimensional systems, but also in 3 - dimensional space models. However, ten years ago [41] it was proved that explicit solutions exist only in low-dimensional systems. The reason of this emergence is that the discrete equation encodes the particle fusion rules in the limit of large values of momentum, reflecting the conformal invariance of the theory. Going over to the limit $k \gg 1$ in discrete equations of motion, the peculiarities of the low-dimensional theory disappear.

The features of considered statistics are summed up by the theory of tensor categories [48, 4]. Coherently this theory unifies the processes of braiding and fusion of the string manifolds, which are images of quasi-particle world lines. The theory of symmetric tensor categories [48, 51, 32] is applicable also to $(3+1)D$ systems. In the last case, restrictions on solutions of equations of motion are so strong, that all anyon states are excluded, and only bosons ($\alpha = 2\pi$) and fermions ($\alpha = \pi$) survive.

To solve the problems of the theory of strongly correlated systems on a lattice, it is often convenient to use the theory of the braid group representations, or the Temperley-Lieb algebra (TLA) representations [14] with a special value of the TLA parameter. In the continuous limit, we can also employ the effective Chern-Simons action [19, 2]. In particular, to classify the hierarchy of phase states described by $(3+0)$-dimensional spinor Ginzburg-Landau functional [3], it is convenient to use the Hopf invariant [42, 43], which is the $(3+0)D$ analog of the Chern-Simons term.

The construction of a lattice model out from the continuous theory (even inheriting its essential properties) is evidently an ambiguous procedure. Some intuitive insight of how this can be done, based on the mentioned properties, may include the following consideration. It is well known that the Chern-Simons term in the action of $(2+1)$-dimensional systems encodes the invariant description of fluctuating Aharonov-Bohm vortices. The action of the doubled Chern-Simons models (for example $SU(N)_k \times \overline{SU}(N)_k$ [32, 19, 2]) describing systems whit time reversal and parity symmetries, include pairs of Aharonov-Bohm vortices with the opposite chiralities. It is natural to suppose such a neutral pair of Aharonov-Bohm vortices as plane slices of a string loop (living in the 3-dimensional space). A pair of Aharonov-Bohm vortices are traces of a loop cut by a plane. This means on the whole that small loops, characterized by a common length scale of the order of the lattice constant, induce dimerized configurations of the currents, when these are projected on the plane. Such a projected loop, or equivalently the dimer configuration, can be a building block for the formation of self-organized mesoscale structures in the form of nets. The increased interest to quantum dimer distributions [2, 33, 23] is connected not only with the theory of resonance valence bonds, but it has a support originating from the exact solvable models [45] and, also, it is motivated by recent results in the field of non-Abelian gauge theory [18].

In this paper we will consider the solutions of nonlinear discrete equations of the thermodynamic Bethe ansatz and will show their relation with the characteristics arising in the approach, based on the use of braid statistics. They are characterized by the quantum dimension, which is obtained as the solution of the mentioned discrete equations. We will give also some arguments in favor of stability of arising mappings of string nets, built of golden chains [13].

2 Discrete equations of exclusion statistics

Let us consider a system, which contains a set, $\{N_a\}$ of particles with types a. The collective index $a = (\alpha, i)$ contains the index α for denoting internal degrees of freedom and index i enumerates rapidities of particles. If we fix the variables of all particles, except the a-th one, the N-particle wave function can be expressed via the one-particle function of the a-th particle. Let D_a be the dimension of such a basis. Then the rate of changing the number of vacant states due to adding N_b particles determines [22] the matrix g_{ab} of statistical interaction in the following way

$$\frac{\partial D_a}{\partial N_b} = -g_{ab}\,. \tag{2.1}$$

Assuming that the matrix g_{ab} does not depend on the set of numbers $\{N_a\}$, we have the solution of the Eq. (2.1):

$$D_a = -\sum_b g_{ab} N_b + D_a^0. \tag{2.2}$$

The Eq. (2.2) contains the number of particles N_b, added to the system, and the number of vacant states D_a^0 of the a-th type in the initial state without particles. The number of holes D_a determines the statistical weight as follows

$$W = \prod_a \frac{(N_a + D_a - 1 + \sum_b g_{ab}\delta_{ab})!}{(N_a)!(D_a - 1 + \sum_b g_{ab}\delta_{ab})!}\,. \tag{2.3}$$

In the cases $g_{ab} = 0$ and $g_{ab} = \delta_{ab}$ the Eq. (2.3) yields well-known statistical weights of Bose and Fermi particles.

The statistical weight W allows to find the entropy $S = \ln W$ and thermodynamical functions. The free energy in the equilibrium state

$$F = -T \sum_a D_a^0 \ln(1 + w_a^{-1}) \tag{2.4}$$

is determined by the function w_a, which can be found from the equation

$$(1+w_a)\prod_b \left(1+w_b^{-1}\right)^{-g_{ab}} = e^{(\varepsilon_a^0-\mu_a)/T} \tag{2.5}$$

The variable w_a can be expressed via so-called pseudo-energies $\varepsilon = T\ln(D_a/N_a)$ by means of the parametrization $w_a = e^{\varepsilon_a/T}$. In Eq. (2.5), T, μ_a and ε_a^0 are the temperature, the chemical potential and the bare energy of quasiparticles of the type a.

We consider the solution of the equation (2.5) in the limit $T \gg \varepsilon_a^0 - \mu_a$. In this case the Eq. (2.5) can be written down in the form

$$w_a = \prod_b \left(1+w_b^{-1}\right)^{N_{ab}} \tag{2.6}$$

which is typical for the thermodynamic Bethe ansatz. Here $N_{ab} = g_{ab} - \delta_{ab}$.

Below we will be interested in the case of ideal statistics [8, 20, 41], when phases of the scattering matrix, being functions of rapidities, have the structure of step functions. In this case the integral equation (2.6) transforms into the algebraic transcendental equation. The matrix N_{ab} can be expressed [44, 25, 11, 37] via the incidence matrix $G_{ab} = \delta_{a+1,b} + \delta_{a,b+1}$ of Lie algebra with the help of the identity $N = G(2-G)^{-1}$. The matrix $2-G$ is the Cartan matrix of the graph A_{k+1}/Z_2 [44]. Using this identity in Eq. (2.6) and replacing $w_a = d_a^2 - 1$ it is easy to see that the Eq. (2.6) has the form

$$d_a^2 = 1 + \prod_{j=1,b=2j}^{[k/2]} d_b^{G_{ab}} = \begin{cases} 1+d_2,\ a=1, \\ 1+d_{a-1}d_{a+1},\ a=2,\ldots,[k/2]-1, \\ 1+d_{[k/2]-1}d_{[k/2]},\ a=[k/2]. \end{cases} \tag{2.7}$$

Here index a is connected with the value of the spin j by the relation $a = 2j$, and the upper limit of the product fixes the Jones-Wenzl projector [14, 19]. We will show in Appendix that the Eq. (2.7) is presented in fact the special limit of the Hirota equation [28].

The distribution function

$$n_a = \frac{1}{d_a^2} = \frac{1}{w_a+1} = \frac{1}{e^{\varepsilon_a/T}+1} \tag{2.8}$$

in our case coincides with the probability $p(a\bar{a} \to 0)$ [40] of annihilation of a particle-antiparticle pair in the system of two linked loops of world lines which describe the process of annihilation of two pairs.

We can find the solution of the Eq. (2.7) taking into account the appropriate boundary conditions by comparing it with the identity

$$[a]_q^2 - 1 = [a+1]_q[a-1]_q\,. \tag{2.9}$$

Here $[a]_q = (q^a - q^{-a})/(q-q^{-1})$, $q = e^{i\pi/(k+2)}$ is the deformation parameter of the $SU(2)_k$ Chern-Simons theory. Identifying $d_a = [a+1]_q$, we can see that solutions of the Eq. (2.7) are quantum dimensions [40, 47, 15, 1, 24]

$$d_a = \frac{\sin[\pi(a+1)/(k+2)]}{\sin[\pi/(k+2)]}\,, \tag{2.10}$$

which are expressed via the Chebyshev polynomials of the second kind, $U_m = \sin[(m+1)\theta]/\sin\theta$ with specification $\theta = \pi/(k+2)$ for A_{k+1} algebra. In the limit $k \gg 1$, $d_{a=2j}$ equals $2j+1$. The meaning of the quantum dimension is as follows. The quantum dimension d_a determines the rate d_a^N at which the dimension of the topological Hilbert space grows after particles are added.

We pay our attention to the fact that the Eq. (2.7) is a fermionic representation [7, 29, 30] of the recursion relation for the Chebyshev polynomials of the second kind., The bosonic representation of recursion relations has the form $U_{m+1}(x) + U_{m-1}(x) - 2xU_m(x) = 0$. From this point of view, we can call the Eq. (4.3) (see Appendix) the anyon representation of recursion relations for the Chebyshev polynomials of the second kind. In result, recursion relations for different representation have following form:

$$P_{s+1}(x) + P_{s-1}(x) - 2xP_s(x) = 0, \;\; \textit{in Bose representation}, \tag{2.11}$$

$$b_m^2 = 1 + b_{m-1}b_{m+1}, \;\; \textit{in Fermi representation}, \tag{2.12}$$

$$Y_t Y_t = (1+Y_{t+1})(1+Y_{t-1}), \;\; \textit{in anyon representation}. \tag{2.13}$$

The roots of the Chebyshev polynomials, being the eigen values of the matrix G, are equal to

$$x_{m,k} = q^{m+1} + q^{-(m+1)} = 2\cos\left(\frac{(m+1)\pi}{k+2}\right). \tag{2.14}$$

The greatest eigen value $x_{0,k}$ of the incidence matrix G is given by the Beraha numbers [6, 34, 5, 35]

$$d = 2\cos[\pi/(k+2)]. \tag{2.15}$$

In particular for the special value $k = 3$, we have the golden ratio $d = (1+\sqrt{5})/2$ which is the solution of the algebraic equation $d^2 = d+1$. For $k = 2$, the Beraha number d and the quantum dimension $d_{2j=1} = \sqrt{2}$ coincide.

To clarify the meaning of the d's one should emphasize that (i) the numbers d determine eigen values of Wilson operator for the contractible unknotted loop [14, 19]. (ii) For special values of the parameter $d = q + q^{-1}$, the generators $B_i = I - qe_i$ satisfy the relations of a braid group under the condition, that the generators e_i satisfy the relation $e_i^2 = de_i$ of the Temperley-Lieb algebra. (iii) The values of the parameter d are nontrivial restriction, which leads to the finite-dimensional Hilbert spaces. (iv) The wavefunction Ψ, defined on the one-dimensional manifold, which is a joining up of the arbitrary tangle α and the Wilson loop $\bigcirc$, i.e. $\Psi(\alpha \cup \bigcirc)$, equals $d\Psi(\alpha)$ [19]. Thus the parameter d has the meaning of the weight of the contractible unknotted Wilson loop, and d^2 acquires the meaning of fugacity [17]. (v) Besides, it turns out, that for the mentioned values of d, the theory is unitary.

Summarizing one can say that the points of intersection of braid statistics and statistics with the generalized exclusion principle are the set of the Beraha points $d = 2\cos[\pi/(k+2)]$ [6, 34, 5, 35], where the processes of braiding and fusion of string manifolds are self-consistently united.

3 Discussion

The problem of the coexistence [2] of locality and braiding can be solved by constructing Hamiltonians $H = \sum_i H_i$ of the Rokhsar-Kivelson (RK) type [45]. Each term $H_i = Q_i^+ Q_i$ in the sum with $Q = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ acts at the RK point as a projector builted by dimer configurations. If we

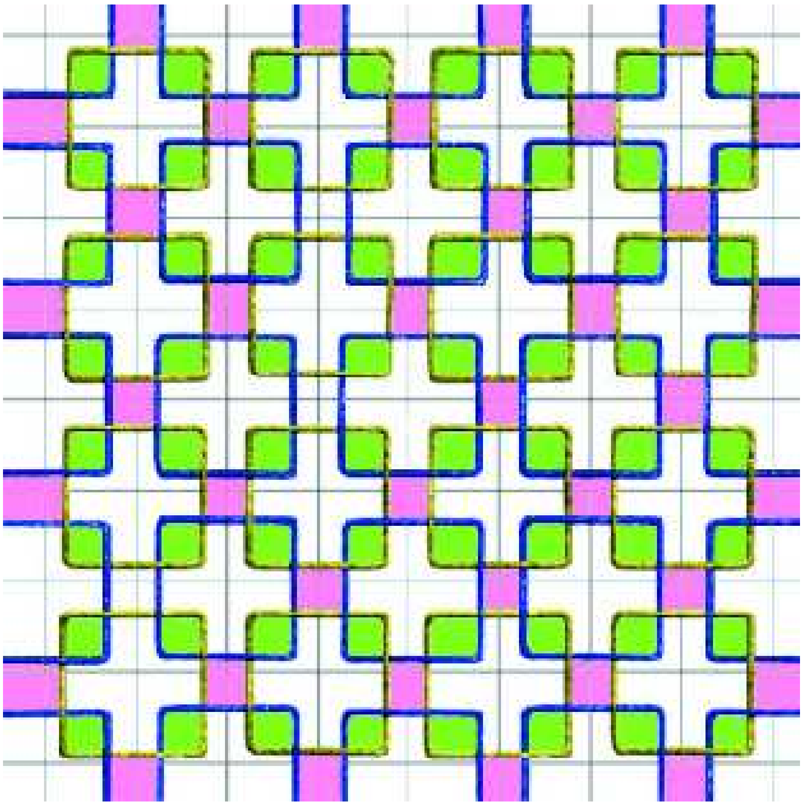

Figure 1: (Color online). A planar representation of the net. Shaded regions show distributions of the fluxes.

locate the dimers on the opposite links of plaquetts we will encounter contradiction due to spatial separation of the braiding phenomena. The best way to solve this problem is the distribution of dimer states between odd and even sites of the lattice [13]. The checkerboard distribution of linked dimer degrees of freedom with defects in the order of effective fluxes (Fig. 2) may be one of the possible ways to solve the problem.

The realistic candidate for the Hamiltonian with such a type of the ground state is the Hamiltonian $H = -\sum_i H_i$ which contains the Temperley-Lieb generators e_i in the form of the projectors

$$H_i = \frac{1}{d} e_i \tag{3.1}$$

to the singlet states. Obviously $H_i^2 = H_i$ due to the Temperley-Lieb commutation relation $e_i^2 = de_i$. The d's here are the Beraha numbers [6, 34, 5, 35] from the second section. Because of the rank-level symmetry, i.e. $SU(N)_k = SU(k)_N$, and the argument based on small values of integers, the $SU(2)_2 \times \overline{SU}(2)_2$ theory is a good candidate. The matrix of the $6j$-symbols [40] in this case is equal to $\pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the braid operator is $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Another important model with computational universal rules is based [13] on a golden chain builted by $SU(2)_3 \times \overline{SU}(2)_3$ Fibonacci anyons.

In summary, we found the quantum dimensions as exact solutions of discrete equations encoding brading and fusion processes. By means of projection of knotted field configurations we proposed the quantum dimer models which incorporated braiding properties of one-dimensional topology.

4 Appendix

Let us show that the Eq. (2.7) is a particular case of the Hirota equation (see seminal papers [26, 28])

$$T_t^a(u+1)T_t^a(u-1) - T_{t+1}^a(u)T_{t-1}^a(u) = T_t^{a+1}(u)T_t^{a-1}(u)\,. \tag{4.1}$$

Here a is the index of the agebra A_{k+1}, t is the discrete time and u is the discrete values of rapidities. The functions $T_t^a(u)$ are the eigen values of the transfer-matrix [28]. For the gauged functions $Y_t^a(u) = T_{t+1}^a(u)T_{t-1}^a(u)/\left(T_t^{a+1}(u)T_t^{a-1}(u)\right)$ the following equation

$$Y_t^a(u+1)Y_t^a(u-1) = \frac{(1+Y_{t+1}^a(u))(1+Y_{t-1}^a(u))}{(1+(Y_t^{a+1}(u))^{-1})(1+(Y_t^{a-1}(u))^{-1})} \tag{4.2}$$

is valid. In the A_1-algebra case the function $Y_t^1(u) \equiv Y_t(u)$ satisfies the equation

$$Y_t(u+1)Y_t(u-1) = (1+Y_{t+1}(u))(1+Y_{t-1}(u)). \tag{4.3}$$

Putting $Y_t = b_t^2 - 1$ in this equation, in the limit $u \gg 1$ we get the equation $b_t^2 = 1 + b_{t+1}b_{t-1}$ which coincides with (2.7). We see that the function Y_t equals w_t.

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