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Solitary Waves in a Madelung Fluid Description of Derivative NLS Equations

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Abstract

Recently using a Madelung fluid description a connection between envelope-like solutions of NLS-type equations and soliton-like solutions of KdV-type equations was found and investigated by R. Fedele et al. (2002). A similar discussion is given for the class of derivative NLS-type equations. For a motion with stationary profile current velocity the fluid density satisfies generalized stationary Gardner equation, and solitary wave solutions are found. For the completely integrable cases these are compared with existing solutions in literature.

1 Introduction

Eighty years ago Madelung [1] gave a hydrodynamic description of quantum mechanics. Writing the wave function as $\Psi = \sqrt{\rho} e^{\frac{i}{\hbar}\theta}$ the Schrödinger equation (1-D case)

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + mU(x)\Psi$$

is equivalent with the pair of coupled equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{1.1}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) v = \frac{\hbar^2}{2m^2} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} \right) - \frac{\partial U}{\partial x} \tag{1.2}$$

where $\rho = |\Psi|^2$ is the fluid density and $v = \frac{1}{m}\frac{\partial \theta}{\partial x}$ is its current velocity. The first is a continuity equation for the fluid density and the second an Euler equation (equation of motion) for the fluid velocity. The last one contains a force term proportional to the gradient of the "quantum potential",

$\frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}$, also known as Bohm's potential. The interpretation of v as a fluid velocity comes from the previous expression of the continuity equation and from the expression taken by the current density j in this representation

$$j = \frac{\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right) = \rho v$$

The Madelung fluid description of quantum mechanics proved to be an useful approach in a number of applications ranging from stochastic mechanics to quantum cosmology (for a historical review see [2]). In the last decade it was succesfully applied to describe quantum effects in mesoscopic systems, in plasma physics and for discussing quantum aspects of beam dynamics in high intensity accelerators (for many references see [3]).

Recently in a series of papers Fedele et al [4] have used a Madelung fluid description to discuss the following generalized 1-D nonlinear Schrödinger equation (gNLS)

$$i\alpha\frac{\partial\Psi}{\partial t} + \frac{\alpha^2}{2}\frac{\partial^2}{\partial x^2} - U(|\Psi|^2)\Psi = 0 \tag{1.3}$$

Here $U(|\Psi|^2)$ depends only on $|\Psi|^2$. For $U = |\Psi|^2$ (1.3) transformed into the usual NLS equation. Writing $\Psi = \sqrt{\rho}e^{\frac{i}{\alpha}\theta}$, the density ρ and the current velocity $v = \partial\theta/\partial x$ are satisfying the same equations (1.1) and (1.2) respectively, with $\hbar$ replaced by α. By a series of transformations [4] the equation (1.2) is transformed into

$$-\rho\frac{\partial v}{\partial t} + v\frac{\partial\rho}{\partial t} + 2\left[c_0(t) - \int\frac{\partial v}{\partial t}dx\right]\frac{\partial\rho}{\partial x} - \left(\rho\frac{dU}{d\rho} + 2U\right)\frac{\partial\rho}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3} = 0 \tag{1.4}$$

where $c_0(t)$ is an integration constant (it may depend on t). In the case of a motion with a stationary profile current velocity, when both $\rho(x,t), v(x,t)$ are depending only on $\xi = x - u_0 t$, the equation (1.1) is integrating giving

$$v(\xi) = u_0 + \frac{A_0}{\rho(\xi)} \tag{1.5}$$

with A_0 an integration constant, (for instance, solutions vanishing at infinity require that $A_0 \equiv 0$), and the equation (1.4) transforms into

$$\frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} - \left(\rho\frac{dU}{d\rho} + 2U\right)\frac{d\rho}{d\xi} + [2c_0 + u_0^2]\frac{d\rho}{d\xi} = 0 \tag{1.6}$$

which is a generalized stationary KdV equation. Several solitary wave solutions were obtained and discussed in [4] assuming $U(\rho) = q_0\rho^\nu$ (bright, dark and gray solitons). Once a non-negative solution of (1.6) is known the phase $\theta(x,t)$ is given by

$$\theta(x,t) = u_0\xi + A_0\int\frac{d\xi}{\rho(\xi)} - c_0 t + \theta_0 \tag{1.7}$$

and the corresponding solution of gNLS equation is completely determined.

It deserves to mention also the use of Madelung fluid description in discussing another completely integrable NLS-type equation, the so called ”resonant nonlinear Schrödinger equation”.

This has been introduced to study low dimensional gravity models, and appears also in plasma physics (see [5] and the references therein).

In the present paper the same procedure will be used to discuss the class of derivative nonlinear Schrödinger type equations. Two distinct types of such equations will be considered, namely

$$i\alpha\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+i\beta\frac{\partial}{\partial x}(U(|\Psi|^2)\Psi)=0 \tag{1.8}$$

called in the followings generalized derivative NLS equation of first kind (gdNLS-1), and

$$i\alpha\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+i\beta U(|\Psi|^2)\frac{\partial\Psi}{\partial x}=0 \tag{1.9}$$

which will be called generalized derivative NLS equation of second kind (gdNLS-2). For $U=|\Psi|^2$ they become completely integrable equations (denoted by dNLS-1 and dNLS-2 respectively), namely

$$i\alpha\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+i\beta\frac{\partial}{\partial x}(|\Psi|^2\Psi)=0 \tag{1.10}$$

and

$$i\alpha\frac{\partial\Psi}{\partial t}+\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2}+i\beta|\Psi|^2\frac{\partial\Psi}{\partial x}=0 \tag{1.11}$$

Especially the dNLS-1 equation is well known in plasma physics. It describes the evolution of small but finite amplitude Alphén waves propagating quasiparallel to a magnetic field in a low β-plasma [6]. Recently the same equation was found to describe the behaviour of large-amplitude magnetohydrodynamic waves, propagating in an arbitrary direction with respect to the magnetic field, in a high β-plasma [7]. Also in nonlinear optics for propagating of very short pulses the typical Kerr nonlinearity has to be supplemented with a derivative term [8].

The dNLS-1 equation (1.10) is a completely integrable system and was solved by IST method by Kaup and Newell [9] for vanishing boundary conditions and by Kawata and Inoue [10] for nonvanishing condition [11]. Alternative methods can be used to find N-soliton solutions of dNLS equations. We mention Hirota's bilinear formalism [12], Darboux transformation technique [13], or the approach of Bäcklund transformations [14]. Periodic solutions of NLS-type equations are carefully investigated by Kamchatnov [15].

In the next section the basic equations describing the gdNLS equations (1.8) and (1.9) in Madelung's fluid description will be derived. In section 3, the solitary solutions vanishing at infinity will be calculated in the case of $U=\rho^\nu$, with $\nu>0$. In the particular case of completely integrable equation dNLS-1 (1.10) a comparison with the known solutions will be done. In section 4, periodic solutions of dNLS-1 and dNLS-2 will be determined using this approach. Finally, remarks and conclusions are given in section 5.

2 Basic equations

In the Madelung fluid representation we write $\Psi=\sqrt{\rho}e^{\frac{i}{\alpha}\theta}$. Introducing it in equation (1.8) and (1.9) and separating the real and the imaginary part, one obtains the continuity equations

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v+\frac{\beta}{\alpha}G(\rho))=0 \tag{2.1}$$

for the fluid density ρ and the equation of motion for the fluid velocity $v = \frac{\partial\theta}{\partial x}$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v = \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) - \frac{\beta}{\alpha}\frac{\partial}{\partial x}(vU(\rho)) \tag{2.2}$$

In (2.1) $G(\rho)$ is defined by

$$\frac{dG}{d\rho} = U + 2\rho\frac{dU}{d\rho} \tag{2.3}$$

and

$$\frac{dG}{d\rho} = U \tag{2.4}$$

for gdNLS-1 and gdNLS-2 respectively. The equation (2.2) is the same for both cases. Following Fedele et al [4] the equation (2.2) is transformed into

$$-\rho\frac{\partial v}{\partial t} + v\frac{\partial\rho}{\partial t} + 2\left[c_0(t) - \int\frac{\partial v}{\partial t}dx\right]\frac{\partial\rho}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3\rho}{\partial x^3} - \frac{\beta}{\alpha}\rho U\frac{\partial v}{\partial x} + \frac{\beta}{\alpha}v\left(-U \pm \rho\frac{dU}{d\rho}\right)\frac{\partial\rho}{\partial x} = 0 \tag{2.5}$$

with (+) sign for gdNLS-1 and (-) for gdNLS-2 respectively. The equations (2.1) and (2.5) are the basic equation for the subsequent discussion.

A first remark showing the difference between the gdNLS and gNLS cases is the following. For gNLS equations a class of solutions (bright, dark solitons) is obtained assuming a constant velocity $v = v_0$ [4]. Then from the continuity equation (1.1) one obtains that the density ρ depends only on the variable $\xi = x - v_0 t$. Here, for the gdNLS, the same assumption should give, by using Eq. (2.1),

$$\frac{\partial\rho}{\partial t} + \left(v_0 + \frac{\beta}{\alpha}\frac{dG}{d\rho}\right)\frac{\partial\rho}{\partial x} = 0 \tag{2.6}$$

which is a dispersionless nonlinear equation whose (implicit) solution is given by

$$\rho(x,t) = f[x - (v_0 + \frac{\beta}{\alpha}\frac{dG}{d\rho})t] \tag{2.7}$$

where $f(x)$ is just the initial condition $f(x) = \rho(x, t = 0)$. This result and the equation (2.6) are incompatible with the dispersive equation (2.5). Therefore a solution with constant velocity is not possible in the case of dNLS-type equations.

3 Stationary profile current velocity

The next choice is a stationary profile current velocity when both $\rho(x,t)$ and $v(x,t)$ depend only on $\xi = x - u_0 t$. Then (2.1) can be integrated giving

$$v = u_0 + \frac{A_0}{\rho} - \frac{\beta}{\alpha}\frac{G(\rho)}{\rho} \tag{3.1}$$

Here A_0 is an integration constant. In this section we consider only the case $U(\rho) = \rho^{\nu}$. From (2.3) and (2.4) we obtain

$$G(\rho) = \frac{2\nu+1}{\nu+1}\rho^{\nu+1} \tag{3.2}$$

for gdNLS-1, and

$$G(\rho) = \frac{1}{\nu+1}\rho^{\nu+1} \tag{3.3}$$

for gdNLS-2. Note that for localized solutions satisfying the boundary conditions

$$\lim_{\xi\to\pm\infty}\rho(\xi) = 0$$

the following conditions for (3.1) are required:

$$A_0 = 0 \qquad \text{and} \qquad \nu > 0.$$

For gdNLS-1 the partial differential equation (2.5) becomes the ordinary differential equation

$$\frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} + (2c_0+u_0^2)\frac{d\rho}{d\xi} - u_0\frac{\beta}{\alpha}(\nu+2)\rho^{\nu}\frac{d\rho}{d\xi} + (\frac{\beta}{\alpha})^2\frac{2\nu+1}{\nu+1}\rho^{2\nu}\frac{d\rho}{d\xi} = 0 \tag{3.4}$$

Integrating twice with ρ, $\frac{d\rho}{d\xi}$ and $\frac{d^2\rho}{d\xi^2}$ vanishing at $|\xi| \to \infty$ one obtains

$$\frac{\alpha^2}{4}\left(\frac{d\rho}{d\xi}\right)^2 = -\rho^2\left[\left(\frac{\beta}{\alpha}\right)^2\frac{1}{(\nu+1)^2}\rho^{2\nu} - 2u_0\frac{\beta}{\alpha}\frac{1}{\nu+1}\rho^{\nu} + (u_0^2+2c_0)\right] \tag{3.5}$$

With the change of variable $z = \frac{1}{\rho\nu}$ it becomes

$$\frac{\alpha^2}{4\nu^2}\left(\frac{dz}{d\xi}\right)^2 = -(u_0^2+2c_0)z^2 + 2u_0\frac{\beta}{\alpha}\frac{1}{\nu+1}z - \left(\frac{\beta}{\alpha}\right)^2\frac{1}{(\nu+1)^2} \tag{3.6}$$

Let us assume first that

$$u_0^2 + 2c_0 = -b^2 < 0 \tag{3.7}$$

Then the second order polynomial in the r.h.s. of (3.6) has two real roots, one positive (z_2) and the other negative (z_1). Also the r.h.s. has to be positive, and because in the asymptotic region $\rho \to 0$ and consequently $z \to \infty$, the region of interest on z-axis is $z \in (z_2, \infty)$. Denoting

$$A = \frac{2\nu}{|\alpha|}b \tag{3.8}$$

the equation (3.6) writes

$$\frac{dz}{d\xi} = A\sqrt{(z-z_1)(z-z_2)} \tag{3.9}$$

Its solution is

$$z(\xi) = z_m + z_M \cosh A\xi \tag{3.10}$$

where

$$z_m = \frac{z_1+z_2}{2} = \frac{1}{b}\frac{|\frac{\beta}{\alpha}|}{\nu+1}(-u_0 \; sign\frac{\beta}{\alpha})$$
$$z_M = \frac{z_2-z_1}{2} = \frac{1}{b}\frac{|\frac{\beta}{\alpha}|}{\nu+1}\sqrt{u_0^2+b^2} \tag{3.11}$$

Then

$$\rho(\xi) = \frac{1}{(z_m + z_M \cosh A\xi)^{\frac{1}{\nu}}} \tag{3.12}$$

If $u_0^2 + 2c_0 > 0$ the second order polynomial in the rhs of (3.6) has complex conjugated roots, and the polynomial is negative everywhere, so this situation is of no interest. It is easily shown that the same equation (3.5) is obtained also in the case of gdNLS-2 equation, although the starting equation (the equivalent of (3.4)) is slightly different.

In the case $\nu = 1$ when gdNLS-1 becomes the completely integrable equation dNLS-1, the equation (3.4) becomes

$$\frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} + (u_0^2+2c_0)\frac{d\rho}{d\xi} - 3u_0\frac{\beta}{\alpha}\rho\frac{d\rho}{d\xi} + \frac{3}{2}\left(\frac{\beta}{\alpha}\right)^2\rho^2\frac{d\rho}{d\xi} = 0 \tag{3.13}$$

which is the stationary Gardner's equation. The solution is

$$\rho = \frac{1}{z_m + z_M \cosh A\xi} \tag{3.14}$$

with $A,\ z_m,\ z_M$ obtained from (3.8) and (3.11) for $\nu = 1$.

In order to calculate the phase $\theta(x,t)$ the expression (3.12) of $\rho(\xi)$ is introduced in (3.1). We get

$$v = \frac{d\theta}{d\xi} = u_0 - \frac{\beta}{\alpha}\frac{2\nu+1}{\nu+1}\frac{1}{z_m + z_M\cosh A\xi} \tag{3.15}$$

which is easily integrated giving

$$\theta(x,t) = u_0\xi - \frac{\beta}{\alpha}\frac{2\nu+1}{\nu+1}\frac{1}{Az_M}\frac{1}{2\sqrt{1-a^2}}\arctan\left[\sqrt{\frac{1-a}{1+a}}\tan\frac{A}{2}\xi\right] - 2c_0t - \theta_0 \tag{3.16}$$

Here we denoted

$$a = \frac{z_m}{z_M} = \frac{-u_0\; sign\frac{\beta}{\alpha}}{\sqrt{u_0^2+b^2}}, \qquad |a| = \frac{u_0}{\sqrt{u_0^2+b^2}} < 1 \tag{3.17}$$

and θ_0 is an initial phase (integration constant).

The results for dNLS-type equation ($\nu = 1$) are similar with those existing in the literature [6], [9], [12]. We remind here the result for the 1-soliton solution obtained by Kaup and Newell using IST method for dNLS-1 equation

$$i\frac{\partial q}{\partial T} + \frac{\partial^2 q}{\partial X^2} + i\frac{\partial}{\partial X}\left(|q|^2 q\right) = 0 \tag{3.18}$$

namely

$$q(X,T) = 4\Delta \sin\, \gamma \frac{e^{-2i\sigma(X,T)}e^{2\theta(X,T)}}{e^{4\theta(X,T)} + e^{-i\gamma}} e^{-2i\mu^+(X,T)}$$

where

$$\theta(X,T) = \eta(X - X_0) - 4\xi\eta T, \qquad \sigma(X,T) = \xi X + 2(\xi^2 - \eta^2)T + \sigma_0$$

$$e^{i\mu^+(X,T)} = \frac{e^{4\theta} + e^{i\gamma}}{e^{4\theta} + e^{-i\gamma}}$$

Here $\zeta = \xi + i\eta$ is the eigenvalue of the spectral problem and $\Delta,\ \gamma$ are defined by

$$\xi = -\Delta^2 \cos\gamma, \qquad \eta = \Delta^2 \sin\gamma$$

It is easily seen that

$$|q|^2 = \frac{8\Delta^2 \sin^2\gamma}{\cos\gamma + \cosh 4\theta} \tag{3.19}$$

is exactly of the form (3.14). The same expression is also found by other authors (see [6], [13], [14])

4 Periodic solutions of dNLS-1

Besides the solitons, another interesting solutions of completely integrable equations are the periodic ones. The problem of finding periodic solutions is known for a long time in mathematical literature. Different methods than in the soliton case have to be used (most used is the ”finite band method”; for a review see [16]), but many times the expressions obtained are rather complicated. Therefore several simplifications were developed to solve the problem in a simpler way. Such an approach was adopted by Kamchatnov and simple expressions for periodic solutions were found for a number of important equations, inclusive the dNLS-1 (see [15], [17] and references therein). Very similar expressions will be found here for dNLS-1 equation using Madelung's fluid description presented in the previous section.

The starting point is Gardner's equation.

$$\frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} + (u_0^2 + 2c_0 + \frac{\beta}{\alpha}A_0)\frac{d\rho}{d\xi} - 3u_0\frac{\beta}{\alpha}\rho\frac{d\rho}{d\xi} + \frac{3}{2}\left(\frac{\beta}{\alpha}\right)^2\rho^2\frac{d\rho}{d\xi} = 0 \tag{4.1}$$

with $A_0 \neq 0$. After twice integrations one obtains

$$\left(\frac{d\rho}{d\xi}\right)^2 = -\left(\frac{\beta}{\alpha}\right)^2 P_4(\rho) \tag{4.2}$$

where $P_4(\rho)$ is a fourth order polynomial in ρ

$$P_4(\rho) = \rho^4 - 4\frac{\beta}{\alpha}u_0\rho^3 + 4\frac{\beta}{\alpha}(u_0^2 + 2c_0 + \frac{\beta}{\alpha}A_0)\rho^2 + B\rho + C \tag{4.3}$$

With B,C other integration constants. We are interested in positive values of ρ for which the rhs of (4.2) is positive also. Keeping this in mind we require that the polynomial $P_4(\rho)$ has at least two positive roots. Let us denote by $\rho_1 > \rho_2 > \rho_3 > \rho_4$ the roots of $P_4(\rho)$ and at least the first two are positive. We list below the interesting situations.

In the case of four real roots with $\rho_1 > \rho_2 > 0$ the solutions of (4.2) for $\rho_2 \leq \rho < \rho_1$ is given by

$$\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(t-\rho_4)(t-\rho_3)(t-\rho_2)(\rho_1-t)}} = \frac{|\beta|}{\alpha^2}\xi \tag{4.4}$$

When all the roots are positive besides (4.4) another interesting situation for $\rho_4 \leq \rho < \rho_3$ is

$$\int_{\rho_4}^{\rho} \frac{dt}{\sqrt{(t-\rho_4)(\rho_3-t)(\rho_2-t)(t-\rho_1)}} = \frac{|\beta|}{\alpha^2}\xi \tag{4.5}$$

It is possible to have two positive real roots ρ_1 and ρ_2 and two complex conjugated. Then

$$\int_{\rho_2}^{\rho} \frac{dt}{\sqrt{(t^2+at+b)(t-\rho_2)(\rho_1-t)}} = \frac{|\beta|}{\alpha^2}\xi \tag{4.6}$$

where $t^2+at+b=0$ has complex roots c and c^*. We mention also the situation when we have four real roots, two of them positive $\rho_1 > \rho_2 >$, and the other two equal $\rho_3 = \rho_4$. Then we get

$$\int_{\rho_2}^{\rho} \frac{dt}{(t-\rho_3)\sqrt{(t-\rho_2)(\rho_1-t)}} = \frac{|\beta|}{\alpha^2}\xi \tag{4.7}$$

In the cases (4.4)-(4.6) the integral in the rhs is an elliptic integral and the solutions can be expressed through Jacobi elliptic functions [18]. Indeed for (4.4) the l.h.s. is given by $gF(\phi,k) = gu$, ([18], formula 256.00), where

$$\begin{aligned}
&k^2 = \frac{(\rho_1-\rho_2)(\rho_3-\rho_4)}{(\rho_1-\rho_3)(\rho_3-\rho_4)}, \quad \mu^2 = \frac{\rho_1-\rho_2}{\rho_1-\rho_3}, \quad k^2 < \mu^2 < 1 \\
&g = \frac{2}{\sqrt{(\rho_1-\rho_3)(\rho_2-\rho_4)}}, \qquad u = \frac{1}{g}\frac{|\beta|}{\alpha^2}\xi \\
&\sin^2\phi = sn^2 u = \frac{(\rho_1-\rho_3)(\rho-\rho_2)}{(\rho_1-\rho_2)(\rho-\rho_3)}
\end{aligned} \tag{4.8}$$

from which we get

$$\rho = \frac{\rho_2 - \rho_3\mu^2 sn^2 u}{1-\mu^2 sn^2 u} \tag{4.9}$$

Here $F(\phi,k)$ is the elliptic integral of first kind of modulus k, and $sn(u,k)$ is Jacobi elliptic sinus of amplitude u and modulus k. From (4.9) we see that $\rho = \rho_2$ for $u = 0$ and $\rho = \rho_1$ for $\rho = K(k)$, $K(k)$ being the complete elliptic integral of first kind.

The integral (4.5) is of the same form, but with other definitions only for μ and $\sin\phi$ (see [18], formula 252.00)

$$\sin^2\phi = sn^2 u = \frac{(\rho_1-\rho_3)(\rho-\rho_4)}{(\rho_3-\rho_4)(\rho_1-\rho)}, \qquad \mu^2 = \frac{\rho_3-\rho_4}{\rho_1-\rho_3} \tag{4.10}$$

Then

$$\rho = \frac{\rho_4+\rho_1\mu^2 sn^2 u}{1-\mu^2\, sn^2 u} \tag{4.11}$$

When we have two complex roots $c,\ c^*$ the result is (see [18] formula 259.00)

$$\cos\phi = cn\ u = \frac{(\rho_1-\rho)B-(\rho-\rho_2)A}{(\rho_1-\rho)B+(\rho-\rho_2)A} \tag{4.12}$$

where

$$\begin{gathered}
A^2 = (\rho_1-b_1)^2+a_1^2 \qquad B^2 = (\rho_2-b_1)^2+a_1^2 \\
a_1^2 = -\frac{1}{4}(c-c^*)^2 \qquad b_1 = \frac{1}{2}(c+c^*) \\
g = \frac{1}{\sqrt{AB}} \\
k^2 = \frac{(\rho_1-\rho_2)^2-(A-B)^2}{4AB} \qquad u = \sqrt{AB}\frac{|\beta|}{\alpha^2}\xi
\end{gathered} \tag{4.13}$$

From (4.12) one obtains

$$\rho = \frac{A\rho_2+B\rho_1+(A\rho_2-B\rho_1)cn\ u}{(A+B)+(A-B)cn\ u} \tag{4.14}$$

and $\rho=\rho_2$ for $u=0$ and $\rho=\rho_1$ for $u=2K(k)$. In the case of (4.7) the result is a rather complicate expression with trigonometric functions which we shall not present here.

In order to calculate the phase $\theta(x,t)$ we start from

$$v = \frac{d\theta}{d\xi} = u_0+\frac{A}{\rho}-\frac{3}{2}\frac{\beta}{\alpha}\rho \tag{4.15}$$

For ρ given by (4.11) one obtains

$$\theta(x,t) = (u_0+\frac{A_0}{\rho_3}-\frac{3}{2}\frac{\beta}{\alpha}\rho_3)\xi-\frac{\rho_2-\rho_3}{\rho_3}A_0\int\frac{d\xi}{\rho_2-\rho_3\mu^2 sn^2 u}-\frac{3}{2}\frac{\beta}{\alpha}(\rho_2-\rho_3)\int\frac{d\xi}{1-\mu^2 sn^2 u}-\theta_0$$

As $d\xi = g\frac{\alpha^2}{|\beta|}du$, and using the integral ([18], formula 363.02)

$$\int\frac{du}{1-\eta sn\ u} = \frac{u}{2}+\frac{1}{2(1-\eta)}\arctan\left[(1-\eta)\frac{sn\ u}{cn\ u\ dn\ u}\right] \tag{4.16}$$

where $\eta = \frac{\rho_3}{\rho}\mu^2$ in the first integral and $\eta=\mu^2$ in the second (in both cases $\eta<1$) finally we obtain

$$\theta(x,t) = \left[u_0+\frac{A_0}{2}\left(\frac{1}{\rho_2}+\frac{1}{\rho_3}\right)-\frac{3}{4}\frac{\beta}{\alpha}(\rho_2-\rho_3)\right]\xi-\theta_0$$

$$-\frac{A_0}{2\rho_3}\frac{\alpha^2}{|\beta|}g\left(1-\frac{\rho_3}{\rho_2}\right)\arctan\left[\left(1-\frac{\rho_3}{\rho_2}\mu^2\right)\frac{sn\ u}{cn\ u\ dn\ u}\right] \quad (4.17)$$
$$-\frac{3}{4}|\alpha|sign\frac{\beta}{\alpha}(\rho_2-\rho_3)g\frac{1}{1-\mu^2}\arctan\left[(1-\mu^2)\frac{sn\ u}{cn\ u\ dn\ u}\right]$$

As is easily seen the function $\phi(u) = (1-\eta)\frac{sn\ u}{cn\ u\ dn\ u}$ is a periodic function of period $2K$, vanishing at $u = 0$ and $u = 2K$ and becomming $\pm\infty$ at $u = K$. Therefore as is expected $\arctan\phi(u) \in (0,\pi)$ when $u \in (0,2K)$ and $\theta(x,t)$ is well behaved. In the same way the phase $\theta(x,t)$ can be calculated for other expressions of $\rho(\xi)$.

5 Remarks and Conclusions

In the present paper the solitary wave solutions for a class of generalized derivative nonlinear Schrödinger equations (gdNLS eqs.) were investigated in Madelung's fluid description. Explicit solutions for vanishing boundary conditions at infinity were obtained for stationary profile current velocity. For arbitrary space and time dependence of $\rho(x,t)$ and $v(x,t)$, when these functions satisfy the coupled set of nonlinear equations (2.1) and (2.5), the problem is still open. In the case of the derivative NLS-1 equation (1.10), using this formalism, the 1-periodic solutions were determined. As expected they are expressible through Jacobi elliptic functions and very similar with those existing in literature. Extension to other gdNLS equations are under way. They are not straightforward because in the rhs of equation (4.2) a higher order polynomial in ρ should appear and the integration leads to hyperelliptic functions. Remarkably, conclusion is that Madelung's fluid description is an useful approach to find special classes of solutions of nonlinear evolution equations.

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