Journal of Nonlinear Mathematical Physics Volume 15, Supplement 3 (2008), 437–448 ARTICLE

Combination of Inverse Spectral Transform Method and Method of Characteristics: Deformed Pohlmeyer Equation

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Abstract

We apply a version of the dressing method to a system of four-dimensional nonlinear Partial Differential Equations (PDEs), which contains both Pohlmeyer equation (i.e. nonlinear PDE integrable by the Inverse Spectral Transform Method) and nonlinear matrix PDE integrable by the method of characteristics as particular reductions. Some other reductions are suggested.

1 Introduction

In this paper we apply a properly modified version of the dressing method developed in [1] to a system of nonlinear Partial Differential Equations (PDEs) which combines some properties of both nonlinear PDEs integrable by the Inverse Spectral Transform Method (ISTM) (or S-integrable PDEs) [2, 3, 4, 5] and nonlinear PDEs integrable by the method of characteristics [6, 7, 8]. The important feature of this version is that it is based on the integral operator with nontrivial kernel [8, 9], unlikely the classical $\bar\partial$-dressing method [4, 5, 10]. The system of nonlinear PDEs studied below can be written as a system of evolution equations,

$$\begin{aligned}
&w_t + u_{x_1} p + w_{x_1} w = 0,\\
&p_t + v_{x_1} p + p_{x_1} w = 0,\\
&(u_t - w u_{x_1} - u v_{x_1})_{x_2} = (u_{x_3} - w u_{x_2} - u v_{x_2})_{x_1},\\
&(v_t - p u_{x_1} - v v_{x_1})_{x_2} = (v_{x_3} - p u_{x_2} - v v_{x_2})_{x_1},
\end{aligned} \tag{1.1}$$

supplemented by the pair of compatible constraints

$$\begin{aligned}
&w_{x_3} + u_{x_2} p + w_{x_2} w = 0,\\
&p_{x_3} + v_{x_2} p + p_{x_2} w = 0,
\end{aligned} \tag{1.2}$$

where fields w, p, v and u are $Q \times Q$ matrices. This system reduces to Pohlmeyer equation [11, 12] (which is S-integrable PDE) if $p = u = w = 0$,

$$v_{tx_2} - v_{x_3x_1} = [v_{x_2}, v_{x_1}] \quad \Leftrightarrow \quad (J^{-1}J_t)_{x_2} - (J^{-1}J_{x_3})_{x_1} = 0 \tag{1.3}$$

(where $v_{x_1} = J^{-1}J_t$, $v_{x_2} = J^{-1}J_{x_3}$) and to to the pair of compatible matrix first order quasilinear PDEs integrable by the method of characteristics if $v = p = u = 0$, [7]:

$$w_t + w_{x_1}w = 0, \quad w_{x_3} + w_{x_2}w = 0. \tag{1.4}$$

An interesting reductions of the system (1.1,1.2) is the following (1+1)-dimensional system:

$$\begin{aligned} &w_t + u_{x_1}p + w_{x_1}w = 0, \quad p_t + v_{x_1}p + p_{x_1}w = 0, \\ &u_t - wu_{x_1} - uv_{x_1} = 0, \quad v_t - pu_{x_1} - vv_{x_1} = 0. \end{aligned} \tag{1.5}$$

A distinguished feature of this system is evident in the case of scalar fields w, p, u and v ($Q = 1$), when eqs.(1.5) read:

$$\vec{w}_t + V\vec{w}_{x_1} = 0, \;\; \vec{w} = \begin{bmatrix} w \\ p \\ u \\ v \end{bmatrix}, \;\; V = \begin{bmatrix} w & 0 & p & 0 \\ 0 & w & 0 & p \\ 0 & 0 & -w & -u \\ 0 & 0 & -p & -v \end{bmatrix} \tag{1.6}$$

In general, 4×4 matrix of this system has 3 different eigenvalues. Thus, this system is intermediate between PDEs integrable by the generalized hodograph method [13] (which requires four different eigenvalues) and method of characteristics for matrix equations [7] (which requires two different eigenvalues).

We will show that solution space to the system (1.1,1.2) is implicitly described by the system of integral-algebraic equations which mixtures integral equation of the classical $\bar\partial$-problem and algebraic equations typical for the method of characteristics [7, 8]. In particular cases, this integral-algebraic system becomes system of algebraic equations, which is quite equivalent to the system derived in [1], see also [7, 8]. According to [1], this fact demonstrates that our nonlinear PDEs possess solutions with wave profile breaking.

In the next section, Sec.2, we represent derivation of the system (1.1,1.2) and its reduction (1.5) by the dressing method. In Sec.3 we describe solution space to the system (1.1,1.2) and give some remarks on the construction of solutions to the eq.(1.5). Conclusions are given in Sec.4.

2 Dressing method: derivation of nonlinear PDEs

2.1 Dressing and spectral functions

In this subsection we introduce basic functions and operators of the dressing algorithm.

Homogeneous Fredholm equation and general form of the spectral system. We start with the following integral equation [8]:

$$\int_D \Psi(\lambda,\nu;x)U(\nu,\mu;x)d\nu \equiv \Psi(\lambda,\nu;x) * U(\nu,\mu;x) = 0, \tag{2.1}$$

where λ and ν are complex (either scalar or vector) spectral parameters, "$*$" means integration over some region D of the spectral parameter space, $x = (x_1, x_2, \dots, t_1, t_2 \dots)$ is a set of all independent variables of nonlinear PDEs; U is $2Q\times Q$ matrix spectral function depending on two spectral parameters; Ψ is $Q\times 2Q$ dressing function and kernel of the integral operator. Following the strategy of [8, 1], we assume that the general solution to eq.(2.1) may be represented in the next form:

$$U(\lambda,\mu;x) = U^{(h)}(\lambda,\nu;x) * f(\nu,\mu;x), \tag{2.2}$$

where $f(\nu,\mu;x)$ is an arbitrary $Q\times Q$ matrix function of arguments and $U^{(h)}$ is a particular nontrivial solution to the homogeneous equation (2.1). This assumption causes the unique linear relation among any two independent solutions $U^{(j)}$, $j=0,1,\dots$, of eq.(2.1):

$$U^{(j)}(\lambda,\mu;x) = U^{(0)}(\lambda,\nu;x) * F^{(j)}(\nu,\mu;x), \tag{2.3}$$

where $F^{(j)}$ are some $Q\times Q$ matrix functions. As we shell see, all solutions $U^{(j)}$ are expressed in terms of the single solution U through some linear operators $L^{(j)}$, either differential or non-differential: $U^{(j)}(\lambda,\mu;x) = L^{(j)}(\lambda,\nu) * U(\nu,\mu;x)$. Thus, eqs.(2.3) represent the general form of the overdetermined compatible system of linear equations for the spectral function U (general form of the spectral system). Besides, we will show in Sec.2.2 that $F^{(j)}$ may be expressed in terms of U using an external $Q\times 2Q$ dressing matrix function $G(\lambda,\mu;x)$, similar to [8, 1].

***x*-dependence of the dressing function Ψ.** We introduce the $Q\times Q$ matrix function $\mathscr{A}(\lambda,\mu)$ and $2Q\times 2Q$ matrix function $A(\lambda,\mu)$ (both functions are independent on x) by the following generalized commutation relation:

$$\mathscr{A}(\lambda,\nu) * \Psi(\nu,\mu;x) = \Psi(\lambda,\nu;x) * A(\nu,\mu) \tag{2.4}$$

and define operators $\mathscr{A}^j$ and A^j as follows: $\mathscr{A}^j = \underbrace{\mathscr{A} * \cdots * \mathscr{A}}_{j}$, $A^j = \underbrace{A * \cdots * A}_{j}$. Let x-dependence of Ψ be given by the equation

$$\Psi_{t_m}(\lambda,\mu;x) + \mathscr{A}(\lambda,\nu) * \Psi_{x_m}(\nu,\mu;x) = 0, \tag{2.5}$$

which is compatible with eq.(2.4).

External dressing function G. We have to introduce an external dressing $Q\times 2Q$ matrix function $G(\lambda,\mu;x)$ which was mentioned above and whose prescription will be explored in Sec.2.2. Let G be defined by the next compatible system of linear equations:

$$G(\lambda,\nu;x) * A(\nu,\mu) = \hat{A}(\lambda,\nu) * G(\nu,\mu;x) + \sum_{j=1}^{2} H_1^{(j)}(\lambda;x) H_2^{(j)}(\mu;x), \tag{2.6}$$

$$G_{t_m}(\lambda,\mu;x) + G_{x_m}(\lambda,\nu;x) * A(\nu,\mu) = 0, \tag{2.7}$$

where $\hat{A}$ and $H_1^{(j)}$, $j=1,2$, are $Q\times Q$, while $H_2^{(j)}$, $j=1,2$, are $Q\times 2Q$ matrix functions. We refer to functions $H_i^{(j)}(\lambda;x)$, $i,j=1,2$ as external dressing functions as well. The compatibility condition of eqs.(2.6) and (2.7) yields:

$$\begin{aligned}&\sum_{j=1}^{2}\Big[\Big(H_1^{(j)}(\lambda;x)\Big)_{t_m} H_2^{(j)}(\mu;x) + {H_1^{(j)}}_{x_m}(\lambda;x) H_2^{(j)}(\nu;x) * A(\nu,\mu) + \\ &H_1^{(j)}(\lambda;x)\Big({H_2^{(j)}}_{t_m}(\mu;x) + {H_2^{(j)}}_{x_m}(\nu;x) * A(\nu,\mu)\Big)\Big] = 0,\end{aligned} \tag{2.8}$$

which admits the following solution:

$$H^{(j)}_{1\ t_m} = H^{(j)}_{1\ x_j} = 0, \ \ j = 1,2, \tag{2.9}$$

$$H^{(j)}_{2\ t_m} + H^{(j)}_{2\ x_m} * A = 0, \ \ j = 1,2, \tag{2.10}$$

i.e. functions $H_1^{(j)}$ do not depend on x: $H_1^{(j)}(\lambda;x) \equiv H_1^{(j)}(\lambda)$, $j = 1,2$. In order to derive PDEs different from the classical S-integrable systems we require

$$\hat{A} * H_1^{(1)} = 0, \ \ \hat{A} * H_1^{(2)} \neq 0. \tag{2.11}$$

2.2 Spectral system for $U(\lambda,\mu;x)$

Now we are ready to derive the overdetermined linear system for the spectral function $U(\lambda,\mu;x)$, i.e. the spectral system. Following the usual strategy of the dressing methods, we have to obtain set of different solutions to the homogeneous eq.(2.1) expressed in terms of functions U, A and x-derivatives of U. For this purpose we apply $\mathscr{A}^m *$ and $(\partial_{t_m} + \mathscr{A} * \partial_{x_m})$ to (2.1) and use eqs.(2.4,2.5). One gets

$$\Psi(\lambda,\nu;x) * E^{(j;m)}(\nu,\mu;x) = 0, \ \ j = 1,2, \tag{2.12}$$

where

$$\begin{aligned} &E^{(1;m)}(\lambda,\mu;x) = A^m(\lambda,\nu) * U(\nu,\mu;x), \\ &E^{(2;m)}(\lambda,\mu;x) = U_{t_m}(\lambda,\mu;x) + A(\lambda,\nu) * U_{x_m}(\nu,\mu;x). \end{aligned} \tag{2.13}$$

Remember the eq.(2.3) relating any two different solutions of the homogeneous equation (2.1). Let $U^{(0)} \equiv U$ in the eq.(2.3) and consider $E^{(j;m)}$ as different solutions of the eq.(2.1). Then eq.(2.3) yields:

$$\begin{aligned} &E^{(1;1)}(\lambda,\mu;x) = U(\lambda,\nu) * \tilde{F}(\nu,\mu;x), \ \Rightarrow \\ &A(\lambda,\nu;x) * U(\nu,\mu;x) = U(\lambda,\nu;x) * \tilde{F}(\nu,\mu;x), \end{aligned} \tag{2.14}$$

$$\begin{aligned} &E^{(2;m)}(\lambda,\mu;x) = U(\lambda,\nu) * F^{(m)}(\nu,\mu;x), \ \Rightarrow \\ &U_{t_m}(\lambda,\mu;x) + A(\lambda,\nu) * U_{x_m}(\nu,\mu;x) = U(\lambda,\nu;x) * F^{(m)}(\nu,\mu;x), \ \ m = 1,2,\ldots. \end{aligned} \tag{2.15}$$

Eqs. (2.14,2.15) represent a preliminary version of the overdetermined linear system for the spectral function $U(\lambda,\mu;x)$.

Recall that $U(\lambda,\mu;x)$ is not unique solution of the integral equation (2.1). To obtain uniqueness we have to introduce one more equation for the spectral function U. For instance, using the external dressing function G, we may write

$$G(\lambda,\nu;x) * U(\nu,\mu;x) = I\delta(\lambda-\mu). \tag{2.16}$$

Now U is *the unique* solution of the system (2.1,2.16). In other words, the equation (2.16) fixes function $f(\lambda,\mu;x)$ in the eq.(2.2). Applying $G*$ to the eqs.(2.14,2.15) and using eq.(2.16) one gets the expressions for $\tilde{F}$ and $F^{(m)}$:

$$\begin{aligned}
\tilde{F}(\nu,\mu;x) &= G(\lambda,\nu;x)*E^{(1;1)}(\nu,\mu;x) = \\
&\hat{A}(\lambda,\mu)+\sum_{j=1}^{2}H_1^{(j)}(\lambda)H_2^{(j)}(\nu;x)*U(\nu,\mu;x), \\
F^{(m)}(\nu,\mu;x) &= G(\lambda,\nu;x)*E^{(2;m)}(\nu,\mu;x) = \\
&\sum_{j=1}^{2}H_1^{(j)}(\lambda)\Big(H_2^{(j)}(\nu;x)*U(\nu,\mu;x)\Big)_{x_m},\ \ m=1,2,\dots.
\end{aligned} \tag{2.17}$$

Although index m may take any positive integer value (reflecting the existence of the hierarchy of commuting flows) we will take only two values $m = 1,2$, which is enough to construct a complete system of nonlinear PDEs. Unlikely the classical spectral systems, eqs.(2.14,2.15) depend on two spectral parameters due to the spectral function $U(\lambda,\mu;x)$. However, functions of single spectral parameter appear in these equations naturally. These functions are following:

$$\begin{aligned}
&V^{(ji)}(\lambda;x) = U(\lambda,\mu;x)*\hat{A}^i(\mu,\nu)*H_1^{(j)}(\nu), \\
&W^{(ji)}(\mu;x) = H_2^{(j)}(\lambda;x)*A^i(\lambda,\nu)*U(\nu,\mu;x),\ \ i,j=1,2.
\end{aligned} \tag{2.18}$$

All in all, substituting eqs.(2.17) and (2.18) into the eqs.(2.14,2.15) one gets:

$$A(\lambda,\nu)*U(\nu,\mu;x) = U(\lambda,\nu;x)*\hat{A}(\nu,\mu)+\sum_{j=1}^{2}V^{(j0)}(\lambda;x)W^{(j0)}(\mu;x), \tag{2.19}$$

$$U_{t_m}(\lambda,\mu;x)+A(\lambda,\nu)*U_{x_m}(\nu,\mu;x) = \sum_{j=1}^{2}V^{(j0)}(\lambda;x)W^{(j0)}_{x_m}(\mu;x),\ \ m=1,2. \tag{2.20}$$

The non-classical type spectral system (2.19,2.20) depending on two spectral parameters λ and μ gives rise to the classical type spectral system for the spectral functions $V^{(j0)}(\lambda;x)$, $j=1,2$ with single spectral parameter. This system appears after applying $*H_1^{(k)}$, $k=1,2$ to the eqs.(2.19) and using eqs.(2.11):

$$\begin{aligned}
&A(\lambda,\nu;x)*V^{(10)}(\nu;x) = \sum_{j=1}^{2}V^{(j0)}(\lambda;x)w^{(j1;00)}(x), \\
&A(\lambda,\nu;x)*V^{(20)}(\nu;x) = V^{(21)}(\lambda;x)+\sum_{j=1}^{2}V^{(j0)}(\lambda;x)w^{(j2;00)}(x), \\
&V^{(k0)}_{t_m}(\lambda;x)+A(\lambda,\nu)*V^{(k0)}_{x_m}(\nu;x)-\sum_{j=1}^{2}V^{(j0)}(\lambda;x)w^{(jk;00)}_{x_m}(x)=0,\ \ k,m=1,2,
\end{aligned} \tag{2.21}$$

where fields $w^{(ij;kn)}$ are defined as follows:

$$w^{(ij;kn)} = H_2^{(i)}*A^k*U*A^n*H_1^{(j)}. \tag{2.22}$$

The definition of fields (2.22) suggests us, for instance, two following reductions:

$$1. \qquad \hat{A}^{n_0}*H_1^{(2)} = \sum_{i=1}^{n_0-1}\hat{A}^i*H_1^{(2)}r^{(i)} \ \ \Rightarrow\ \ w^{(j2;kn_0)} = \sum_{i=1}^{n_0-1}w^{(j2;ki)}r^{(i)},\ \ \forall j,k, \tag{2.23}$$

$$2. \qquad H_2^{(2)}*A^{k_0} = \sum_{i=1}^{k_0-1}r^{(i)}H_2^{(2)}*A^i \ \ \Rightarrow\ \ w^{(2j;k_0n)} = \sum_{i=1}^{k_0-1}r^{(i)}w^{(2j;in)},\ \forall j,n, \tag{2.24}$$

where n_0 and k_0 are any integer numbers and $r^{(i)}$ are scalar parameters.

2.3 Nonlinear PDEs

Applying $H_2^{(i)}*$, $i = 1,2$ to the system (2.21) we obtain the following system of nonlinear PDEs:

$$w^{(i1;10)} = \sum_{j=1}^{2} w^{(ij;00)} w^{(j1;00)}, \tag{2.25}$$

$$w^{(i2;10)} = w^{(i2;01)} + \sum_{j=1}^{2} w^{(ij;00)} w^{(j2;00)}, \tag{2.26}$$

$$E_1^{(ik;00;m)} \quad := \quad w_{t_m}^{(ik;00)} + w_{x_m}^{(ik;10)} - \sum_{j=1}^{2} w^{(ij;00)} w_{x_m}^{(jk;00)} = 0, \;\; i,k,m = 1,2. \tag{2.27}$$

Eliminating $w^{(i1;10)}$ from the eq.(2.27), $k = 1$, using eq.(2.25) one gets

$$w_{t_m}^{(i1;00)} + \sum_{j=1}^{2} w_{x_m}^{(ij;00)} w^{(j1;00)} = 0, \;\; i,m = 1,2. \tag{2.28}$$

Eq.(2.27), $k = 2$ in view of eq.(2.26) may be given another form,

$$E_2^{(i2;00;m)} := w_{t_m}^{(i2;00)} + w_{x_m}^{(i2;01)} + \sum_{j=1}^{2} w_{x_m}^{(ij;00)} w^{(j2;00)} = 0, \;\; i,m = 1,2, \tag{2.29}$$

which is convenient for imposing the reduction (2.23). Now the complete system of nonlinear PDEs is represented by the eqs.(2.28) and the following combination of eqs.(2.27), $k = 2$: $(E_1^{(i2;00;1)})_{x_2} - (E_1^{(i2;00;2)})_{x_1}$. Introducing new dependent and independent variables

$$w = w^{(11;00)}, \;\; p = w^{(21;00)}, \;\; u = w^{(12;00)}, \;\; v = w^{(22;00)}, \;\; t = t_1, \;\; x_3 = t_2, \tag{2.30}$$

we end up with the system (1.1,1.2).

Reductions. Reduction (2.23) yelds quasilinear matrix first order PDEs integrable by the method of characteristics [7]. Similar PDEs have been considered in [1] as lower dimensional reductions of appropriate Self-dual type S-integrable PDEs. It was shown that such lower dimensional PDEs generate solutions with wave profile breaking. A new type of reductions is represented by the eq.(2.24). In the simplest example $k_0 = 1$, the eqs.(2.27) and (2.28) with $m = 1$ yield the system (1.5).

3 Implicit description of solutions to nonlinear PDEs

We introduce the next block-matrix representation of the functions, $\forall\, i, j$:

$$\Psi(\lambda,\mu;x) = [\psi_0(\lambda,\mu;x) \;\; \psi_1(\lambda,\mu;x)], \qquad U(\lambda,\mu;x) = \begin{bmatrix} u_0(\lambda,\mu;x) \\ u_1(\lambda,\mu;x) \end{bmatrix}, \tag{3.1}$$

$$V^{(ij)}(\lambda;x) = \begin{bmatrix} v_0^{(ij)}(\lambda;x) \\ v_1^{(ij)}(\lambda;x) \end{bmatrix}, \qquad W^{(ij)}(\mu;x) = w_0^{(ij)}(\mu;x),$$

$$G(\lambda,\mu;x) = [g_0(\lambda,\mu;x) \;\; g_1(\lambda,\mu;x)], \qquad H_2^{(i)}(\lambda;x) = [h_{20}^{(i)}(\lambda;x) \;\; h_{21}^{(i)}(\lambda;x)],$$

$$H_1^{(i)}(\lambda;x) = h_1^{(i)}(\lambda;x).$$

Any function in the RHS of the formulae (3.1) is $Q\times Q$ matrix function, so that Ψ, G and $H_2^{(i)}$ are $Q\times 2Q$, U and $V^{(ij)}$ are $2Q\times Q$, $W^{(ij)}$ and $H_1^{(i)}$ are $Q\times Q$ matrix functions. We also have to fix the functions $\mathscr{A}(\lambda,\mu)$, $A(\lambda,\mu)$ and $\hat{A}(\lambda,\mu)$:

$$\mathscr{A}(\lambda,\mu)=\hat{A}(\lambda,\mu)=\lambda\delta(\lambda-\mu)I,\;\; A(\lambda,\mu)=\lambda\delta(\lambda-\mu)I_2, \tag{3.2}$$

where I and I_2 are $Q\times Q$ and $2Q\times 2Q$ identity matrices respectively. Eq.(2.4) suggests us the following structure of Ψ:

$$\Psi(\lambda,\mu;x)=\hat{\Psi}(\lambda;x)\delta(\lambda-\mu),\;\; \hat{\Psi}(\lambda;x)=[\hat{\psi}_0(\lambda;x)\;\;\hat{\psi}_1(\lambda;x)], \tag{3.3}$$

where $\hat{\psi}_i$, $i=0,1$, are $Q\times Q$ matrix functions.

The main feature of the spectral system (2.19,2.20) is the presence of the spectral equation which has no derivatives with respect to x, see eq.(2.19), which suggests us the next representation for U:

$$\begin{aligned}&U(\lambda,\mu;x)=\frac{\sum_{j=1}^{2}V^{(j0)}(\lambda;x)W^{(j0)}(\mu;x)}{\lambda-\mu}+U_0(\lambda;x)\delta(\lambda-\mu), \\ &U_0(\lambda;x)=\left[\begin{array}{c}u_{00}(\lambda;x)\\ u_{01}(\lambda;x)\end{array}\right],\end{aligned} \tag{3.4}$$

where u_{0i}, $i=0,1$, are $Q\times Q$ matrix functions. Remark, that eq.(2.19) after applying $*H_1^{(1)}$ yields:

$$V^{(10)}(\lambda;x)(\lambda I-w^{(11;00)}(x))=V^{(20)}(\lambda;x)w^{(21;00)}(x), \tag{3.5}$$

which relates spectral functions $V^{(10)}$ and $V^{(20)}$. Let us write this relation for the case of diagonalizable $w^{(11;00)}$, i.e.

$$w^{(11;00)}(x)=P(x)E(x)P^{-1}(x), \tag{3.6}$$

$$P_{\alpha\alpha}=1, \tag{3.7}$$

where E is the diagonal matrix of eigenvalues, P is the matrix of eigenvectors with normalization (3.7). Then, multiplying eq.(3.5) by $P(\lambda I-E(x))^{-1}$ from the right one gets

$$\begin{aligned}&V^{(10)}(\lambda;x)P=V^{(20)}(\lambda;x)w^{(21)}(x)(\lambda I-E(x))^{-1}+\hat{V}(\lambda;x)\delta(\lambda I-E(x)), \\ &\hat{V}=\left[\begin{array}{c}\hat{v}_0\\ \hat{v}_1\end{array}\right],\;\; w^{(21)}=w^{(21;00)}P.\end{aligned} \tag{3.8}$$

Similarly, eq.(2.6) yelds

$$\begin{aligned}&G(\lambda,\mu;x)=-\sum_{j=1}^{2}\frac{H_1^{(j)}(\lambda;x)H_2^{(j)}(\mu;x)}{\lambda-\mu}+G_0(\lambda;x)\delta(\lambda-\mu), \\ &G_0(\lambda;x)=[g_{00}(\lambda;x)\;\;g_{01}(\lambda;x)],\end{aligned} \tag{3.9}$$

where g_{0i}, $i=0,1$, are $Q\times Q$ matrix functions and $G_0(\lambda;x)\delta(\lambda-\mu)$ is a solution of the eq.(2.7).

Next, after substitution the eqs.(3.4) and (3.9) into the eq.(2.16), one gets:

$$\frac{1}{\lambda-\mu}\sum_{j=1}^{2}\Big[E_{1j}(\lambda;x)W^{(j0)}(\mu;x)+H_1^{(j)}(\lambda;x)E_{2j}(\mu;x)\Big]+ \tag{3.10}$$
$$G_0(\lambda;x)U_0(\lambda;x)\delta(\lambda-\mu)=I\delta(\lambda-\mu),$$
$$E_{1j}(\lambda;x)=G_0(\lambda;x)V^{(j0)}(\lambda;x)-$$
$$\int_D d\nu\frac{\sum_{i=1}^2 H_1^{(i)}(\lambda;x)H_2^{(i)}(\nu;x)V^{(j0)}(\nu;x)}{\lambda-\nu}-H_1^{(j)}(\lambda;x),$$
$$E_{2j}(\mu;x)=-\int_D d\nu\frac{H_2^{(j)}(\nu;x)\sum_{i=1}^2 V^{(i0)}(\nu;x)W^{(i0)}(\mu;x)}{\nu-\mu}-$$
$$H_2^{(j)}(\mu;x)U_0(\mu;x)+W^{(j0)}(\mu;x).$$

The eq.(3.10) must be identity for any λ and μ. Thus, it must be splited into the following set of equations:

$$G_0(\lambda;x)U_0(\lambda;x)=I, \tag{3.11}$$
$$E_{1j}(\lambda;x)=0, \tag{3.12}$$
$$E_{2j}(\mu;x)=0. \tag{3.13}$$

The last terms in the expressions E_{2j} have been introduced in order to eqs.(3.13) coincide with eq.(3.4) after applying $H_2^{(j)}*$ to eq.(3.4), which is necessary condition. The last terms in the expressions for E_{1j} are needed to compensate the last terms of E_{2j} in the eq.(3.10).

The system (3.13) may be viewed as $2Q^2$ scalar equations for $2Q^2$ elements of the matrix functions $W^{(j0)}$, $j=1,2$, i.e. $W^{(j0)}$ are completely defined. However, eqs.(3.11,3.12) are not a complete system for U_0 and $V^{(j0)}$, $j=1,2$. In fact, eq.(3.11) represents Q^2 scalar equations for $2Q^2$ elements of the matrix function U_0. Similarly, eq.(3.12) represents $2Q^2$ scalar equations for $4Q^2$ elements of the matrix functions $V^{(j0)}$, $j=1,2$. Thus, both eq.(3.11) and eq.(3.12) are underdetermined systems.

The rest of equations for the elements of $V^{(j0)}$ and U_0 follows from the eq.(2.1) after substitution the eq.(3.3) for Ψ and the eq.(3.4) for U:

$$\hat\Psi(\lambda;x)\frac{\sum_{j=1}^2 V^{(j0)}(\lambda;x)W^{(j0)}(\mu;x)}{\lambda-\mu}+\hat\Psi(\lambda;x)U_0(\lambda;x)\delta(\lambda-\mu)=0. \tag{3.14}$$

Since eq.(3.14) must be identity for any λ and μ, it is equivalent to the following equations for $V^{(j0)}$ and U_0:

$$\hat\Psi(\lambda;x)V^{(j0)}(\lambda;x)=0,\ \ j=1,2\ \stackrel{(3.8)}{\Rightarrow}\ \hat\Psi(\lambda;x)\hat V(\lambda;x)=0, \tag{3.15}$$
$$\hat\Psi(\lambda;x)U_0(\lambda;x)=0 \tag{3.16}$$

Each of these matrix equations represents Q^2 scalar equations for $2Q^2$ elements of one of the matrix functions $V^{(j0)}$, $j=1,2$ and U_0. Thus, the system (3.11-3.13,3.15,3.16) is the complete system for elements of $V^{(j0)}(\lambda;x)$, $W^{(j0)}(\mu;x)$, $j=1,2$ and $U_0(\lambda;x)$. Having these functions, the spectral function $U(\lambda,\mu;x)$ may be constructed using the formula (3.4).

It is simple to observe, that essentially important for construction of $w^{(ij;00)}=H_2^{(i)}*V^{(j0)}$ are eqs.(3.12) and (3.15) defining $V^{(j0)}$. The following Proposition is valid.

Proposition 1. *If reductions (2.23) and (2.24) have not been involved into consideration, then*

1. *Eqs.(3.12) and (3.15) are equivalent to the next system of integral-algebraic equations:*

$$v_0^{(20)}(\lambda;x) - \int_D d\nu \frac{\chi^{(2)}(\nu;x)v_0^{(20)}(\nu;x)}{\lambda-\nu} + \delta(\lambda)w_0(x) = I, \tag{3.17}$$

$$w_0(x) = \int_D d\nu \, \frac{\chi^{(1)}(\nu;x)v_0^{(20)}(\nu;x)}{\nu},$$

$$\int_D d\nu \, \chi^{(1)}(\nu;x)v_0^{(20)}(\nu;x)w^{(21;00)}(x)(\nu I - w^{(11;00)}(x))^{-1} = w^{(11;00)}(x), \tag{3.18}$$

where

$$\chi^{(j)}(\lambda;x) = \int_{\mathbb{R}^N} dq \, e^{I\sum_{m=1}^{2} q_m(x_m-\lambda t_m)} \chi_0^{(j)}(\lambda,q), \;\; j=1,2. \tag{3.19}$$

Here $\chi_0^{(j)}$, $j=1,2$ are arbitrary $Q\times Q$ matrix functions, $q=(q_1,\dots,q_N)$, parameters q_i are complex in general.

2. *Expressions for fields $w^{(j2;00)}$, $j=1,2$, and $w^{(21;00)}$ follow from the definition (2.22):*

$$w^{(21;00)}(x) = \chi^{(2)}(\lambda;x) * v_0^{(10)}(\lambda;x), \tag{3.20a}$$

$$w^{(j2;00)}(x) = \chi^{(j)}(\lambda;x) * v_0^{(20)}(\lambda;x), \;\; j=1,2, \tag{3.20b}$$

where

$$v_0^{(10)}(\lambda;x) = v_0^{(20)}(\lambda;x)w^{(21)}(x)(\lambda I - w^{(11;00)}(x))^{-1} \tag{3.21}$$

3. *Eqs.(3.17,3.18,3.20) represent the complete system of integral-algebraic equations which defines fields $w^{(ij;00)}$, $i,j=1,2$.*

Proof. To satisfy the eqs.(2.9) and (2.11) with $\hat{A}$ given by the first of eqs.(3.2) we take

$$H_1^{(1)}(\lambda;x) = \delta(\lambda)I, \;\; H_1^{(2)}(\lambda) = I. \tag{3.22}$$

Note that this form of $H_1^{(2)}$ may be used unless reduction (2.23) is involved. Then eqs.(3.15) and (3.16) yield respectively

$$v_1^{(j0)}(\lambda;x) = -\hat{\psi}_1^{-1}(\lambda;x)\hat{\psi}_0(\lambda;x)v_0^{(j0)}(\lambda;x), \;\; j=1,2, \tag{3.23}$$

$$\hat{v}_1(\lambda;x) = -\hat{\psi}_1^{-1}(\lambda;x)\hat{\psi}_0(\lambda;x)\hat{v}_0(\lambda;x), \tag{3.24}$$

$$u_{01}(\lambda;x) = -\hat{\psi}_1^{-1}(\lambda;x)\hat{\psi}_0(\lambda;x)u_{00}(\lambda;x). \tag{3.25}$$

Thus, eq.(3.12), $j=2$, gets the next form:

$$\phi(\lambda;x)v_0^{(20)}(\lambda;x) - \int_D d\nu \frac{\Big(\chi^{(2)}(\nu;x) + \delta(\lambda)\chi^{(1)}(\nu;x)\Big)v_0^{(20)}(\nu;x)}{\lambda-\nu} = I, \tag{3.26}$$

where

$$\phi(\lambda;x) = g_{00}(\lambda;x) - g_{01}(\lambda;x)\hat{\psi}_1^{-1}(\lambda;x)\hat{\psi}_0(\lambda;x), \tag{3.27}$$
$$\chi^{(j)}(\lambda;x) = h_{20}^{(j)}(\lambda;x) - h_{21}^{(j)}(\lambda;x)\hat{\psi}_1^{-1}(\lambda;x)\hat{\psi}_0(\lambda;x), \quad j = 1,2.$$

Requiring $\phi(\lambda;x) = I$ we get eq.(3.17). Function $\chi^{(2)}$ must provide uniqueness of $v_0^{(20)}$ as a solution of eq.(3.17). Classical $\bar{\partial}$-problem for Pohlmeyer equation corresponds to $\chi^{(1)} = 0$ (and, as a consequence, $w_0 = 0$) [15, 16].

Multiplying eq.(3.12), $j = 1$, by P from the right, substituting eq.(3.8) for $V^{(10)}P$, using eqs.(3.23) with $j = 1$ and eq.(3.17) for $v_0^{(20)}(\lambda;x)$ we obtain:

$$\hat{v}_0(\lambda;x)\delta(\lambda I - E) + \delta(\lambda)\left[\int_D d\nu\, \chi^{(1)}(\nu;x)v_0^{(20)}(\nu;x)w^{(21)}(x)(\nu I - E)^{-1}E^{-1} + \right. \tag{3.28}$$
$$\left. \int_D d\nu\, \frac{\chi^{(1)}(\nu;x)\hat{v}_0(\nu;x)}{\nu}\delta(\nu I - E) - P(x)\right] = 0,$$

which is equivalent to the next pair of equations:

$$(\hat{v}_0(E_\beta;x))_{\alpha\beta} = 0, \quad \alpha,\beta = 1,\ldots,Q, \tag{3.29}$$
$$\int_D d\nu\, \chi^{(1)}(\nu;x)v_0^{(20)}(\nu;x)w^{(21)}(x)(\nu I - E(x))^{-1}E^{-1}(x) = P(x). \tag{3.30}$$

The matrix equation (3.30) is a system of scalar equations for the elements of E and P. We may replace the matrices E and P by $w^{(11;00)}$ in the eq.(3.30). For this purpose we multiply eq.(3.30) by EP from the right resulting in the eq.(3.18).

Thus, we have derived an integral equation (3.17) for $v_0^{(20)}$ and algebraic eq.(3.18) as an equation relating fields $w^{(i1;00)}$, $i = 1,2$. One more equation relating these two fields follows from the eqs.(2.22) (with $i = 2$, $j = 1$, $k = n = 0$) and may be written as eq.(3.20a). Eq.(3.21) follows from the eq.(3.8) after applying P^{-1} from the right and using eqs.(3.23) and (3.29). Two more fields $w^{(i2;00)}$, $i = 1,2$, may be calculated using eqs.(2.22) with $i = 1,2$, $j = 2$, $k = n = 0$, see eqs.(3.20b).

Functions $\chi^{(j)}$, $j = 1,2$, satisfy the linear PDEs which follow from the linear PDEs for the functions $\hat{\psi}_i$ and $h_{2i}^{(j)}$, $i = 0,1$. These PDEs are eqs.(2.5) and (2.10), which may be written as a single linear PDE

$$\varphi_{t_m}(\lambda;x) + \lambda\varphi_{x_m}(\lambda;x) = 0, \quad m = 1,2, \tag{3.31}$$

where φ is one of the functions $\hat{\psi}_i$ or $h_{2i}^{(j)}$, $i = 0,1$ $j = 1,2$. This means that $\chi^{(j)}$ are solutions of the same PDE as well, i.e. $\chi^{(j)}$ may be written in the form (3.19). ■

Reductions. There is a remarkable sub-manifold of particular solutions corresponding to the reduction (2.23). System of integral-algebraic equations (3.17,3.18,3.20) will be replaced by the system of algebraic equations. In the simplest case $n_0 = 1$, $A(\lambda,\mu) = \lambda\delta(\lambda-\mu)$, $\hat{A}(\lambda,\mu) = \lambda(\lambda - a)\delta(\lambda-\mu)$, $H_1^{(1)}(\lambda) = \delta(\lambda)I$, $H_1^{(2)}(\lambda) = \delta(\lambda - a)I$, $a = const$. Then eq.(2.19) after applying $*H_1^{(1)}$ and $*H_1^{(2)}$ yields:

$$\lambda\mathbf{V}(\lambda;x) = \mathbf{V}(\lambda;x)w(x), \quad \mathbf{V} = [V^{(1)} \quad V^{(2)}], \quad w = \begin{bmatrix} w^{(11;00)} & w^{(12;00)} \\ w^{(21;00)} & w^{(22;00)} \end{bmatrix}. \tag{3.32}$$

In result we will get the next algebraic system implicitly describing some family of solutions:

$$w_{\alpha\beta} = \sum_{\gamma=1}^{Q} \left[F_{\alpha\gamma}(x_1 - wt_1, x_2 - wt_2)\right]_{\gamma\beta}, \quad \alpha, \beta = 1, \dots, Q, \tag{3.33}$$

which may be derived by the algebraic method [7]. Here $F(z_1, z_2)$ is arbitrary $Q \times Q$ matrix function.

The second reduction, eq.(2.24), is assotiated with more complicated form of functions $A(\lambda, \mu)$, $\hat{A}(\lambda, \mu)$, $H_1^{(1)}(\lambda)$ and $H_2^{(2)}(\lambda)$. The possible choice might be two-component spectral parameter $\lambda = (\lambda_1, \lambda_2)$, $A(\lambda, \mu) = \lambda_1 \delta(\lambda_1 - \mu_1)\delta(\lambda_2 - \mu_2)$, $\hat{A}(\lambda, \mu) = \lambda_2 \delta(\lambda_1 - \mu_1)\delta(\lambda_2 - \mu_2)$, $H_1^{(1)}(\lambda) = \delta(\lambda_2) I$, $H_2^{(2)}(\lambda; x) = \delta(\lambda_1) I$. However, we postpone the detailed study of this reduction.

4 Conclusions

We represent a simplest example of nonlinear PDEs which may be treated by a version of the dressing method and admits reductions to Self-dual type *S*-integrable PDEs as well as to PDEs integrable by the method of characteristics. Remember, that similar joining of *S*- and *C*-integrable models (see [17] for definition of *C*-integrability) has been represented in [14]. In [8] we have considered a version of the dressing method joining *C*-integrability and integrability by the method of characteristics. It is remarkable, that system (1.1,1.2) admits reduction to the system (1.5), which may not be referred to neither PDEs integrable by the generalized hodograph method nor to PDEs integrable by the method of characteristics. All these examples demonstrate flexibility of the dressing method.

Acknowledgments. The work was supported by RFBR grants 04-01-00508, 06-01-90840, 06-01-92053 and grant NS 7550.2006.2.

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