Stability Research on Boundary Values of Quasilinear Elliptic Equation

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Abstract: In recent years, the research on partial differential equation attracts more and more scholars' attention; the quasilinear elliptic equation is an important part of partial differential equation field, and the stability research on boundary values of quasilinear elliptic equation can be established on the basis of existence and uniqueness. Through appropriate research and discussion on existence and uniqueness of boundary values of quasilinear elliptic equation, this paper carries out a research on stability of boundary values of quasilinear elliptic equation under condition of region, P value, and capacity, etc.

Key words: quasilinear elliptic equation; boundary values of equation; stability

With the progress of the era and social development, more and more scholars expand the research on partial differential equation, especially research on quasilinear elliptic equation. Because the quasilinear elliptic equation is an important part of partial differential equation field, it is closely related with medical science, economics, industry, biology, and engineering science, and it is often applied in information science, physics, and other disciplines and fields. As for quasilinear elliptic equation, the stability of boundary values is established on the existence and uniqueness of boundary values, thus this paper carries out an appropriate research on stability of quasilinear elliptic equation under the condition of region, and index P, etc.

I. Existence and uniqueness of boundary values of quasilinear elliptic equation

In order to carry out research and discussion on existence and uniqueness of boundary values of quasilinear elliptic equation, we firstly make use of and have the aid of the theorem of implicit function to discuss the following (1.1) equation set formulas for the purpose of proving that the boundary values of quasilinear elliptic equation have

continuous dependence and further proving the existence and uniqueness of boundary values of quasilinear elliptic equation.

$$\begin{cases} (r^{n-1}H(u'(r)))'+r^{n-1}f(r,(r(r)))=0, \ r\geq 0 \\ u(r)\rangle 0, \ r\geq 0 \\ u(0)=\alpha\rangle 0, \ u'(0)=0 \end{cases} \quad (1.1)$$

According to the formula (1.1), it is able to obtain the equation

$$u(r)=\alpha-\int_0^r H^{-1}(\int_0^r (\frac{\sigma}{s})^{n-1}\int(\sigma,\ u(\sigma))\ ds$$ (1.2), thus it is able to carry out the research

on existence and uniqueness via the equation in formula (1.2), wherein H (t) and f (x, u) are set to meet following conditions:

1. $f:[0,\ \infty)\times(0,\ \infty)\to(0,\ \infty)$ is continuous;

2. $H^{-1}(t)$ is continuous and differentiable, and f is first-order continuous and partial;

3. $H(0)=H^{-1}(0)=0$;

4. Upon u>0, f (x, u) has consistent local continuity;

5. $H^{-1}(t)$ has local continuity;

6. H (t) belongs to strict increase;

7. In f (t, s), t is strictly monotonic decreasing for s.

Theorem 1: Given that the condition 1-5 are satisfied, it is able to obtain the local solution in existence and uniqueness of equation (1.2) as u (r, a) and rϵ[0, r_α].

Through using the formula (2) as auxiliary function, it is able to obtain following equation:

$$F(r,\ u)=u(r)-\alpha+\int_0^r H^{-1}(\int_0^r (\frac{\sigma}{s})^{n-1}f(\sigma,\ u(\sigma))\ d\sigma)\ ds$$

At this time, given that $r_0=0$, $u_0=\alpha$, and the point p_0 (α, β), it is able to find that the following results can be established.

1. F (r, u) is continuous within region U on one domain of the point p_0;

2. F (p_0) =0;

3. Within region U, F is first-order continuous and partial function;

4. F_u (0, α) =1$\neq$0

According to the theorem of implicit function, it is able to find that the unique definition can be determined in formula (2), that is, the implicit function contained in the region of domain U (r_0) in one region upon $r_0=0$, thus we can find that the solution of equation set (1) is positively local, that is to say, we can obtain as below:

u （r， α)$\in C$ （0， r_α)$\cap$[0， r_α]: through calculation with Gronwall inequation, it is able to obtain u_1 (r) =u_2 (r), r∈[0, r_0], thus we can find that the positive local solution of formula (2) has existence and uniqueness.

Theorem 2: Given that the condition 1, 3, 4, 6, and 7 are satisfied, both α and β are positive constant, and α>β, it is able to obtain 0>u'_α>u'_β, u_α>u_β, r∈[0， R_β], R_α≥R_β. Because f (r, u) belongs to positive function, it is able to obtain:

$$H(u'_\alpha (r)) = -\int_0^r (\frac{\sigma}{r})^{n-1} f（\sigma, u_\alpha) d\sigma \langle 0,$$

Through condition 3 and 6, it is able to obtain u'_α (r) <0, and also obtain u'_β (r) <0.

II. Stability of boundary values of quasilinear elliptic equation

2.1 Stability of weak-solution boundary values within region of quasilinear elliptic equation

Through discussion and research on the quasilinear elliptic equation

$$\begin{cases} -\Delta_p u = a （x)|u|^{\mu-2}u + \lambda|u|^{\alpha-1}|\upsilon|^{\beta+1}u + f_1(x) \\ -\Delta_q v = b （x)|v|^{\upsilon-2}v + \lambda|u|^{\alpha-1}|v|^{\beta-1}v + f_2(x) \\ x \in \Omega \end{cases}(1)$$

, it is found that the following boundary value demand shall be met: u=gk (x), v=hk (x), x∈∂Ω (2); in formula (2), given that k=1，2，3，……，Ω⊂N (N≥3) and given that it is a ∂Ω bounded region with smooth boundary, the (u, v) in equation set in formula (1) belongs to Sobolev space, where $W^{1,p}$ (Ω) $\times W^{1,q}$ (Ω) and 2<p≤q<N; the p, q conjugate number of the function f_1 (x) ∈$L^{p'}$ (Ω), f_2 (x) ∈$L^{q'}$ (Ω) are expressed as $p'=\dfrac{p}{p-1}$， $q'=\dfrac{q}{q-1}$ respectively; as for a (x), b (x) ∈L∞ (Ω), 1<μ<υ<p, the positive constant shall meet the condition of $\dfrac{\alpha+1}{p}+\dfrac{\beta+1}{q}=1$; λ belongs to a positive parameter.

Meanwhile, given that L is a positive constant with appropriate size, its boundary

function (gk (x), hk (x))must meet following conditions: $(g_k(x)$, $h_k(x) \in W^{1-\frac{1}{P},P}(\partial\Omega) \times W^{1-\frac{1}{q},q}(\partial\Omega)$, and $\|$(gk (x), hk (x))$\| \le L.W^{1,p}(\Omega)$. According to space embedding theorem, K_1 is constant, thus $\|u\|_p \le K_1\|u\|_{1,p}$ as for any $u \in W^{1,p}(\Omega)$. According to same principle, it is able to obtain embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$, and then obtain the best constant K_2.

T_1 and T_2 are set as the continuous mapping produced from $W^{1-\frac{1}{P},P}(\partial\Omega)$, $W^{1-\frac{1}{q},q}(\partial\Omega)$ to $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ to make the boundary function g and h locate on the trace and realize $T_1g|_{\partial\Omega}=g$ and $T_2h|_{\partial\Omega}=h$. Furthermore, there are 2 positive constants M_1 and M_2; under the condition of $(T_1g, T_2h) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, and $\|T_1g\| \le M_1\|g\|$, $\|T_2h\| \le M_2\|h\|$, with the help of mapping mode, it is able to obtain $u=y+T_1g_k$, $v=z+T_2h_k$, thus it is able to transform formula (1) and (2) into homogeneous boundary value problem, and then carry out definition solution for variation structure on equation set and energy generic function of $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ on the space, that is, meeting the critical point of formula (3) shown as below:

$$I(y, z) = \frac{\alpha+1}{p}\int\left|\nabla_y + \nabla T_1g_k\right|^p dx +$$
$$\frac{\beta+1}{q}\int\left|\nabla_z + \nabla h_k\right|^q dx -$$
$$\lambda\int\left|y + T_1g_k\right|^{\alpha+1}\left|z + T_2h_k\right|^{\beta+1} dx -$$
$$\frac{\alpha+1}{\mu}\int a(x)\left|y + T_1g_k\right|^{\mu} dx -$$
$$(\alpha+1)\int f_1(x)(y + T_1g_k) dx -$$
$$\frac{\beta+1}{\upsilon}\int b(x)\left|z + T_2h_k\right|^{\upsilon} dx -$$
$$(\beta+1)\int f_2(x)(z + T_2h_k) dx$$

It is given that the function (y, z) of $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ in Banach space has zero boundary value, and the norm of function is defined as $\|(y, z)\| = \max\left\{\|y\|_{1,p}, \|z\|_{1,q}\right\}$, with $\|u\|_{1,p} = \left[\int|\nabla u|^p dx\right]^{\frac{1}{p}}$.

Then, it is given that $(g_k,\ h_k)$, $k \in N$ is a sequence of function which belongs to trace space; therefore, any for any function set in trace space: $(\phi,\ \varphi) \in W_0^{1,\ p}(\Omega) \times W_0^{1,\ q}(\Omega)$, it is able to obtain following functional sequence:

$M_k(y,\ z)\ (\phi,\ \varphi) =$

$\int \left|\nabla\ (y + T_1 g_k)\right|^{p-2} \nabla\ (y + T_1 g_k)\ \nabla \phi dx +$

$\int \left|\nabla\ (z + T_2 h_k)\right|^{q-2} \nabla\ (z + T_2 h_k)\ \nabla \varphi dx$

$N_K(y,\ z)\ (\phi,\ \varphi) =$

$\int a\ (x) \left|y + T_1 g_k\right|^{\mu-2} (y + T_1 gk)\ \phi dx +$

$\int b\ (x) \left|z + T_2 h_k\right|^{\upsilon-2} (z + T_2 h_k)\ \varphi dx +$

$\lambda \int \left|y + T_1 g_k\right|^{\alpha-1} \left|z + T_2 h_k\right|^{\beta+1} \times$ (4)

$(y + T_1 g_k)\ \phi dx + \lambda \int \left|y + T_1 g_k\right|^{\alpha+1} \times$

$\left|z + T_2 h_k\right|^{\beta-1} (z + T_1 h_k)\ \phi dx +$

$\int f_1(x)\ \phi dx + \int f_2(x)\ \varphi dx$

Then, while the generic function I (y, z) belongs to $W_0^{1,\ p}(\Omega) \times W_0^{1,q}(\Omega)$, all critical points (y, z) shall meet $M_k(y,\ z)\ (\phi,\ \varphi) - N_k(y,\ z)\ (\phi,\ \varphi) = 0(5)$; in the formula (5), the left end is set as $J_k(y,\ z)\ (\phi,\ \varphi)$, and the set of function set (y, z) set in formula (5) is U_k, and then the weak-solution set owned by quasilinear elliptic equation set (1) and (2) is $V_k = U_k + (T_1 g_k,\ T_2 h_k)$ (6).

Through effective connection among formula (1), (2), (3), (4), (5), and (6) in above 3.1, it is able to obtain following theorem conclusions.

Theorem 1: Given that the sequence $(g_k,\ z_k)$ of boundary function can have convergence in trace space, while it is up to $(g_0,\ h_0)$ via strong convergence, the sequence $J_k(y,\ z)\ (\phi,\ \varphi)$ produced on Br will have uniform convergence to $J_0(y,\ z)\ (\phi,\ \varphi)$.

Theorem 2: In case of any $(y,\ z)$, $(\phi,\ \varphi) \in B_r$, there will be a uniform convergence for functional sequence $J_k(y,\ z)\ (\phi,\ \varphi)$ on Br; while it is up to

$J_0(y, z)$ (ϕ, φ) via convergence, the following two points will exist: firstly, as for each k=1, 2, 3, …, the set U_k is compact at the significance of norm topology; secondly, the sequence $\{(y_k, z_k)\}$ which meets $(y_k, z_k) \in U_k$, $k \in N$ is compact, that is to say, at the significance of strong topology, lim sup$U_k \subset U_0$, which means that $U_k \to U_0$ is satisfied.

2.2 Stability of quasilinear elliptic equation at p index

Given that Ω belongs to bounded open set in R^n (n>3) region and the boundary it has belongs to consistent p thickness, it is able to obtain according to quasilinear elliptic equation:

$$\begin{cases} -\text{div} A_p(x, u, Du) = f(x), & \text{in}\Omega \\ u \in W_0^{1-p}(\Omega), 1\langle p\langle\infty \end{cases} \quad (1)$$

Given that $A_p: \Omega \times R \times R^n \to R^n$ is a function belongs to Carathédory in (M_1), as for any $s \in R$, $\xi \in R^n$, a.e.x∈Ω, it is required to meet $A_p(x, s, \xi \geq \alpha|\xi|^p$, $\forall \xi \in R^n$

(2) and $|A_p(x, s, \xi)| \leq b_1(x) + b_2(x)|s|^q + |\xi|^{p-1}$ (3); upon $\xi_1 \neq \xi_2$, there will be $(A_p(x, s_1, \xi_1,) - A_p(x, s_2, \xi_2)) (\xi_1 - \xi_2)\rangle 0$; the formula (3) and (4) can change, and A_p changes with the change of p.

Meanwhile, given that $Ai(x, u_i, \nabla u_i) = Ap_i(x, u_i, \nabla u_i)$, the problem way of Dirichlet shall be considered according to equation demand, and then it is able to determine the stability of weak-solution p index on quasilinear elliptic equation set via the equation in formula (5).

$$\begin{cases} -\text{div} A_i(x, u, Du) = f(x), & \text{in}\Omega \\ u_i \in W_0^{1-p}(\Omega), 1\langle p_i\langle\infty \end{cases} \quad (5)$$

Some hypotheses can be made as for $A_i(x, s, \xi)$ and Ω within the region. It is given that (S_1) is a sequence in (p_i), and then we can set $1\langle p_i\langle\infty$, $p_i \to p_0$; as for any i=0, 1, 2, 3, …, the condition of $(M_1)(p=p_i)$ shall be met in case of $A_i = A_{pi}(x, s, \xi)$.

a.e.x $\in \Omega$, $A_i(x, s, \xi) \to A_0(x, s, \xi)$ is uniform convergence made by R^n on a compact subset; besides, it is able to assume (S$_2$) as below:

$$b_1(x), \ f(x) \in L^\mu(\Omega), \ \mu \rangle \frac{(n+p_0)p_0}{n(p_0-1)}, \ b_2(x) \in L^\infty(\Omega), 1\langle q\langle p_i - 1$$

However, (S$_3$) Ω^e belongs to consistent p thickness.

Theorem: If (M$_1$), (S$_1$), (S$_2$), (S$_3$) are established, $ui \in W_0^{1, p_i}(\Omega)$ is a weak solution which belongs to formula (5), and then there will be t>p$_0$; in $W_0^{1, t}(\Omega)$, u_i is transformed to via norm convergence u$_0$; t tightly depends on n, p$_0$, α, β and c$_0$.

III. Stability of opposite boundary values of quasilinear elliptic equation

The quasilinear elliptic equation is assumed as below:

$$\begin{cases} -\Delta_p u = \lambda Q(x)|u|^{p-2}u, x \in \Omega, \\ u = gk(x), x \in \partial\Omega, k = 0,1,2,3,\ldots\ldots \end{cases} \quad (3.1)$$

Upon p$\geq$2, k=0, 1, 2, 3, ……, the weak solution set $U_k \in E \cap L^\infty(\Omega)$ obtained via research and discussion on quasilinear elliptic equation (3.1) belongs to set convergence and has certain continuous dependence on boundary value gk$\in$W of quasilinear elliptic equation; in case of 1<p$\leq$2, there exists the unique weak solution $U_k \in E \cap L^\infty(\Omega)$ as for quasilinear elliptic equation (3.1), thus there is certain point convergence function as for stability of boundary values of quasilinear elliptic equation.

While the set $V_0 \subset E_0$ is V$_k$ on set sequence, under the condition of the upper limit of k=1, 2, 3, … and the condition that y$\in$V$_0$ is a lower-limit point of strong topological significance on E$_0$ in the sequence $\{yk\}$ of function yk$\in$V$_k$ (k=1, 2, 3, …) at any point of V$_0$, the limit on V$_k$ can be marked as lim supV$_k\approx$V$_0$. In other words, the set U$_k$ is located in E$_0$ and it can be converged into the set U$_0$. Given that lim sup$U_k \subset U_0$ is established, it is obvious that the set convergence is equal to point convergence in E$_0$ while U$_k$ and U$_0$ belong to single-point set.

Given that g$_k$ (k=1, 2, 3, …) belongs to the sequence of a boundary function in W, as

for any $\varphi \in E_0$, it is able to define the functional sequence owned in E_0 as: $J_k(y) = \langle A_k(y), \phi \rangle - \langle B_k(y), \phi \rangle$, and then the weak-solution set of quasilinear elliptic equation can be expressed into $U_k = \{y \in E_0 : \forall \phi \in E_0, J_k(y) = 0\}$. According to relevant theorem, as for any $k \in N$, the set U_k belongs to non-emptiness; then, as for all existing k=1, 2, 3, …, there will be constant r>0 not depending on k and there will exist $U_k \subset B(r) \subset E_0$.

If J_k (y) on functional sequence have uniform convergence in B (r) till J_0 (y), the limit set lim sup U_k owned by U_k in solution set sequence belongs to non-emptiness, and such condition is weak closing condition, thus there exists $\lim \sup U_k \subset U_0$ under weak topological significance of E_0.

Besides, through relevant theorem, it is able to find that there exists $u_k \in E$ in at least one solution after the quasilinear elliptic equation (3.1) is used for solution as for any $k \in N$, and the obtained solution set belongs to non-emptiness; if 1<p≤2 appears, it means that $u_k \in E$ has certain unique existence, that is to say, U_k belongs to single-point set. Meanwhile, there is constant R>0 and there exists $u_k \subset B(R)$, and all existing k=1, 2, 3, … can be met, thus it is able to find that U_k in solution set sequence has weak compactness in E. However, if the trace function $g_k \to g_0$ has strong convergence in W, $J_k(\cdot)$ existing in functional sequence can have uniform convergence in closed B (r)$\subset E_0$ till $J_0(\cdot)$. In case of p≥2, according to relevant theorem, it is able to find that there always exists the solution set of $U_k \to U_0$ (k→∞) under the condition of E weak topology and strong topology, that is to say, if the weak solution set U_k in quasilinear elliptic equation (3.1) is proposed before set convergence, it is able to enhance the stability of opposite boundary value $g_k \in W$ of quasilinear elliptic equation. In case of 1<p≤2, the stability of quasilinear elliptic equation is of point convergence significance.

IV. Conclusion

In a word, as the research and application on partial differential equation attract more and more scholars' attention in recent years, there are also high achievements on this

aspect. On the basis of research and discussion on existence and uniqueness of boundary values of quasilinear elliptic equation, this paper carries out a research and discussion on stability of boundary values of quasilinear elliptic equation in terms of region and P value, and proves that the boundary values of quasilinear elliptic equation have continuous dependence at significance of weak and strong convergence, thus ensuring the stability of boundary values of quasilinear elliptic equation.

Acknowledgments

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