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Analysis of Navier-Stokes Equation from the Viewpoint of Advection Diffusion

Hiroki Sakamoto *, Tetsuo Hattori, Akiomi Tada, Vanhoa Nguyen
Graduate School of Engineering, Kagawa University, 2217-20, Hayashi-Cho
Takamatsu, Kagawa 761-0396, Japan

Hiromichi Kawano
NTT Advanced Technology Corporation, 19-18, Nakamachi
Musashino, Tokyo 180-0006, Japan
E-mail: hiromichi.kawano@ntt-at.co.jp

Abstract

We propose a highly accurate approximate solution for Navier-Stokes Equation (NSE) based on the similarity between NSE and Advection Diffusion Equation (ADE). First, we present the analytical exact solution of ADE using a Green function (integral kernel) that is obtained from the diffusion equation over uniform flow field (or velocity field) in three dimensional (3D) boundless region under arbitrary initial condition. Next, from the explicit similarity between NSE and ADE, we derive the approximate solution of NSE using the analytical solution of ADE.

***Keywords*:** approximate analysis method, Navier-Stokes Equation (NSE), Advection Diffusion Equation (ADE)

1. Introduction

Navier-Stokes Equation (NSE) is well known as fundamental one in Fluid Mechanics[1]. Because the exact analytical solution of NSE is not yet obtained, we have to use numerical computation for the solution under the arbitrary initial and boundary conditions.

For the numerical computation method, Difference Method, Finite Element Method and Boundary Element Method are well known. However, as for the Difference Method, it tends to be difficult to deal with complicated boundary conditions. In addition, the method has to satisfy the well-known Courant condition to obtain the stable computation solution.

On the other hand, the Finite Element Method takes much time to solve simultaneous equations appeared in the method, and the Boundary Element Method has a problem in the computation precision for the analysis of viscous flow of high Reynolds number.

*2217-20 Hayashi Cho Takamatsu City, Kagawa, 761-0396 Japan, correspondence Email: hattori@eng.kagawa-u.ac.jp

Even if we obtain the result by using such numerical computations, those methods take much time and the results are involved by not a little computation error.
So, in order to reach the more accurate solution, it would be desirable to have an analytical approximate solution that is as much as close to the exact one.
In order to obtain such an analytical approximate solution, we focus on the similarity between the NSE and Advection Diffusion Equation (ADE). And, we derive the exact analytical solution of the ADE over uniform flow field (or velocity field) in three dimensional (3D) boundless region under arbitrary initial condition[2-4].
Moreover, we derive the highly accurate approximate solution of NSE, using the solution of ADE.

2. Advection Diffusion Equation (ADE)

Let C be a fluid of density (or density of material), and let D_x, D_y, D_z denote the diffusion coefficient in x, y, z axis direction, respectively. Similarly, let u, v, w denote the flow velocity in x, y, z axis direction, respectively. Moreover, let λ and Q be the attenuation coefficient that is spatially uniform and the load generation rate function, respectively.
The partial differential equation of the ADE is shown as Eq.(1).

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2} - u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial y} - w\frac{\partial C}{\partial z} - \lambda C + Q \qquad \cdots(1)$$

Here we introduce the Dirac's δ function as in the following (2).

$$\begin{cases} (x=0)\ \delta(x)=\infty,\ (x\neq 0)\ \delta(x)=0, \\ \int_{-\infty}^{\infty}\delta(x)dx = 1 \end{cases} \qquad \cdots(2)$$

The initial condition and the load generation rate function are shown in Eq.(3) and Eq.(4), respectively.

$$C(x,y,z,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_0(\xi,\eta,\zeta)\,\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)d\xi d\eta d\zeta \qquad \cdots(3)$$

$$Q(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi,\eta,\zeta,t)\,\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)d\xi d\eta d\zeta \qquad \cdots(4)$$

For the Eq.(1), we have the exact analytical solution[5] as follows.

$$C(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_0(\xi,\eta,\zeta)G(x,y,z,t,\xi,\eta,\zeta,0) + \int_0^t Q(\xi,\eta,\zeta,\tau)G(x,y,z,t,\xi,\eta,\zeta,\tau)d\tau\}d\xi d\eta d\zeta$$

where

$$G(x,y,z,t,\xi,\eta,\zeta,\tau) = \frac{1}{8\sqrt{\pi^3\int_\tau^t D_x ds\int_\tau^t D_y ds\int_\tau^t D_z ds}} \exp\left\{-\frac{\left(x-\xi-\int_\tau^t u ds\right)^2}{4\int_\tau^t D_x ds} - \frac{\left(y-\eta-\int_\tau^t v ds\right)^2}{4\int_\tau^t D_y ds} - \frac{\left(z-\zeta-\int_\tau^t w ds\right)^2}{4\int_\tau^t D_z ds} - \int_\tau^t \lambda ds\right\} \qquad \cdots(5)$$

3. Checking the exact analytical solution of the ADE

In Eq.(1), let $D_x = D_y = D_z = D$. Using the following notation in (6), the Eq.(1) can be represented as Eq.(7).

$$(\mathbf{v}\cdot\nabla) = (v_x, v_y, v_z)\bullet\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right) = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial y} \qquad \cdots(6)$$

$$\frac{\partial C}{\partial t} = D\nabla^2 C - (\mathbf{v}\cdot\nabla)C - \lambda C + Q \qquad \cdots(7)$$

Let $\boldsymbol{r}_p = (x, y, z)$, $\boldsymbol{r} = (\xi, \eta, \zeta)$, where $\boldsymbol{r}_p$ means the position at the present time point and $\boldsymbol{r}$ means the general position at the past time point, respectively. Using the vector notation, Eq.(5) can be represented as shown in Eq.(8).

$$G(x, y, z, t, \xi, \eta, \zeta, \tau) = \frac{1}{8\sqrt{\pi^3 [D(t-\tau)]^3}} \exp\left\{ \frac{-\| \mathbf{Z} \|^2}{2\sigma^2} - \int_\tau^t \lambda ds \right\}$$

where $\bar{\boldsymbol{v}} = (u, v, w)$, $\sigma = \sqrt{2D(t-\tau)}$,

$$K = \left(\sigma\sqrt{\pi}\right)^{-3}, \quad \boldsymbol{Z} = \boldsymbol{r_p} - \left(\boldsymbol{r} + \int_\tau^t \bar{\boldsymbol{v}} \, ds \right) \qquad \cdots(8)$$

In order to check the exact analytical solution of the ADE, we substitute Eq.(5) for Eq.(7). First, we use the Δt as shown in Eq.(9).

$$\Delta t = t - \tau \qquad \cdots(9)$$

Then, K and σ in Eq.(8) can be written as follows.

$$\sigma = \sqrt{2D\Delta t} \ , \ K = \left(\sigma\sqrt{\pi}\right)^{-3} = \{2\pi D \Delta t\}^{-\frac{3}{2}} \qquad \cdots(10)$$

Using Eq. (10), we take the derivative of Eq. (10) with respect to t as shown Eq.(11).

$$\frac{\partial K}{\partial t} == -\frac{3}{2} \cdot K \cdot \left(\frac{1}{\Delta t} \right) \qquad \cdots(11)$$

When we take the derivative of Eq.(8) with respect to t , we only differentiate the Green function (Eq. (8)).

$$\frac{\partial}{\partial t} G(x, y, z, t, \xi, \eta, \zeta, \tau) = K \left[-\frac{3D}{\sigma^2} + \frac{D}{\sigma^4} \left\langle \mathbf{Z} + \frac{\sigma^2}{D} \bar{\mathbf{v}}(t) \middle| \mathbf{Z} \right\rangle - \lambda \right] \exp\left\{ -\frac{\|\mathbf{Z}\|^2}{2\sigma^2} - \lambda \cdot \frac{\sigma^2}{2D} \right\} \qquad \cdots(12)$$

Taking the derivative of left hand side (LHS) in Eq.(7) and using the Eq.(12), we have the Eq.(13).
Next, when we take the derivative of Eq.(8) with respect to x , we only have to differentiate the Green function in Eq.(8).

$$\begin{aligned} \frac{\partial C(x, y, z, t)}{\partial t} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_0(\xi, \eta, \zeta) K_0 \left[-\frac{3D}{{\sigma_0}^2} + \frac{D}{{\sigma_0}^4} \left\langle \mathbf{Z}_0 + \frac{{\sigma_0}^2}{D} \bar{\mathbf{v}}(t) \middle| \mathbf{Z}_0 \right\rangle - \lambda \right] \exp\left\{ -\frac{\|\mathbf{Z}_0\|^2}{2{\sigma_0}^2} - \lambda \cdot \frac{{\sigma_0}^2}{2D} \right\} d\xi d\eta d\zeta \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^t QK \left[-\frac{3D}{\sigma^2} + \frac{D}{\sigma^4} \left\langle \mathbf{Z} + \frac{\sigma^2}{D} \bar{\mathbf{v}}(t) \middle| \mathbf{Z} \right\rangle - \lambda \right] \exp\left\{ -\frac{\|\mathbf{Z}\|^2}{2\sigma^2} - \lambda \cdot \frac{\sigma^2}{2D} \right\} d\tau \right\} d\xi d\eta d\zeta \\ &+ Q(\mathbf{r}, t) \end{aligned}$$

where $\langle \boldsymbol{x} | \boldsymbol{y} \rangle$ means the inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. And,

$$\sigma_0 = \sqrt{2Dt} \ , \ K_0 = \left(\sigma_0\sqrt{\pi}\right)^{-3}, \ \mathbf{Z}_0 = \mathbf{r_p} - \left(\mathbf{r} + \int_0^t \bar{\mathbf{v}} ds \right) \qquad \cdots(13)$$

The first derivative of the Green function is as follows.

$$\frac{\partial}{\partial x} G(x, y, z, t, \xi, \eta, \zeta, \tau) = -\frac{K}{\sigma^2} \left\{ x - \xi - \int_\tau^t u ds \right\} \exp\left\{ -\frac{\left\| \mathbf{r_p} - \mathbf{r} - \int_\tau^t \bar{\mathbf{v}} ds \right\|^2}{2\sigma^2} - \int_\tau^t \lambda ds \right\} \qquad \cdots(14)$$

And, the second derivative is obtained as follows.

$$\frac{\partial^2}{\partial x^2} G(x, y, z, t, \xi, \eta, \zeta, \tau) = -\frac{K}{\sigma^2} \left[1 - \frac{1}{\sigma^2} \left\{ x - \xi - \int_\tau^t u ds \right\}^2 \right] \exp\left\{ -\frac{\left\| \mathbf{r_p} - \mathbf{r} - \int_\tau^t \bar{\mathbf{v}} ds \right\|^2}{2\sigma^2} - \frac{\lambda\sigma^2}{2D} \right\} \qquad \cdots(15)$$

Then, from Eq.(14) and Eq.(15), we have the following Eq.(16) and Eq.(17), respectively.

$$\begin{aligned} \frac{\partial C}{\partial x} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{K_0 C_0(\xi, \eta, \zeta)}{{\sigma_0}^2} \left\{ x - \xi - \int_0^t u ds \right\} \exp\left\{ -\frac{\|\mathbf{Z_0}\|^2}{2{\sigma_0}^2} - \int_0^t \lambda ds \right\} d\xi d\eta d\zeta \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^t -\frac{KQ(\xi, \eta, \zeta, \tau)}{\sigma^2} \left\{ x - \xi - \int_\tau^t u ds \right\} \exp\left\{ -\frac{\|\mathbf{Z}\|^2}{2\sigma^2} - \int_\tau^t \lambda ds \right\} d\tau \right\} d\xi d\eta d\zeta \end{aligned} \qquad \cdots(16)$$

$$\begin{aligned} \frac{\partial^2 C}{\partial x^2} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{K_0 C_0(\xi, \eta, \zeta)}{{\sigma_0}^2} \left[1 - \frac{1}{{\sigma_0}^2} \left\{ x - \xi - \int_0^t u ds \right\}^2 \right] \exp\left\{ -\frac{\|Z_0\|^2}{2{\sigma_0}^2} - \frac{\lambda{\sigma_0}^2}{2D} \right\} d\xi d\eta d\zeta \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^t \frac{KQ(\xi, \eta, \zeta, \tau)}{\sigma^2} \left[1 - \frac{1}{\sigma^2} \left\{ x - \xi - \int_\tau^t u ds \right\}^2 \right] \exp\left\{ -\frac{\|\mathbf{Z}\|^2}{2\sigma^2} - \frac{\lambda\sigma^2}{2D} \right\} d\tau \right\} d\xi d\eta d\zeta \end{aligned} \qquad \cdots(17)$$

In the same way as above, the derivative of the C with respect to y and z can be obtained respectively. Therefore, the right hand side (RHS) of Eq.(7) is represented as Eq.(18).

$$
\begin{aligned}
& D \cdot \nabla^2 C - (\mathbf{v} \cdot \nabla) C - \lambda C + Q \\
& = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0 C_0(\xi,\eta,\zeta) \left[-\frac{3D}{{\sigma_0}^2} + \frac{D}{\sigma_0^4} \left\langle \mathbf{Z_0} + \frac{\sigma_0^2}{D} \bar{\mathbf{v}} \middle| \mathbf{Z_0} \right\rangle - \lambda \right] \exp\left\{ -\frac{\|\mathbf{Z_0}\|^2}{2{\sigma_0}^2} - \frac{\lambda {\sigma_0}^2}{2D} \right\} d\xi d\eta d\zeta \\
& + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^t KQ(\xi,\eta,\zeta,\tau) \cdot \left[-\frac{3D}{\sigma^2} + \frac{D}{\sigma^4} \left\langle \mathbf{Z} + \frac{\sigma^2}{D} \bar{\mathbf{v}} \middle| \mathbf{Z} \right\rangle - \lambda \right] \exp\left\{ -\frac{\|\mathbf{Z}\|^2}{2\sigma^2} - \frac{\lambda \sigma^2}{2D} \right\} d\tau \right\} d\xi d\eta d\zeta \\
& + Q
\end{aligned}
\quad \cdots(18)
$$

Because Eq.(13) (LHS of Eq.(7)) equals Eq.(18) (RHS of Eq.(7)), we have proved that the Eq.(5) is the exact analytical solution of the ADE (Eq.(7)).

4. Navier-Stokes Equation (NSE)

We consider that the NSE shows a Law of conservation of momentum with respect to $\rho v_i \ (i = x, y, z)$ and that NSE is a kind of Advection Diffusion Equation (ADE) with respect to the momentum.

$$\frac{\partial(\rho v_i)}{\partial t} + div(-\mu \nabla v_i + \rho v_i \boldsymbol{v}) = -\frac{\partial p}{\partial x_i} + f_i \qquad \ldots (19)$$

where ρ, v, μ, p, f are density, velocity, coefficient of viscosity, pressure, and external force, respectively.

In NSE (Eq.(19)), putting $\mu = k\rho$ where k is coefficient of kinematic viscosity, we have

$$\frac{\partial \boldsymbol{v}}{\partial t} = k\nabla^2 \boldsymbol{v} - (\boldsymbol{v} \bullet \nabla)\boldsymbol{v} - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \qquad \cdots(20)$$

At $t = t_0$, we have the following relation.

$$\frac{\partial \boldsymbol{v}(t_0)}{\partial t} = k\nabla^2 \boldsymbol{v}(t_0) - \{\boldsymbol{v}(t_0) \bullet \nabla\}\boldsymbol{v}(t_0) - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \qquad \cdots(21)$$

Here, we consider that $\boldsymbol{v}(t_0) = \boldsymbol{v}(t_0 - \Delta t)$, where Δt is infinitesimal time parameter. So we have

$$\frac{\partial \boldsymbol{v}(t_0)}{\partial t} = \boldsymbol{k}\nabla^2 \boldsymbol{v}(t_0) - \{\boldsymbol{v}(t_0 - \Delta t) \bullet \nabla\}\boldsymbol{v}(t_0) - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \qquad \cdots(22)$$

Then putting $\bar{v} = \boldsymbol{v}(t_0 - \Delta t)$, we have another form of the Eq.(7) and Eq.(20), respectively, as follows.

$$\frac{\partial C}{\partial t} = D\nabla^2 C - (\bar{\boldsymbol{v}} \bullet \nabla)C - \lambda C + Q \qquad \cdots(23)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = k\nabla^2 \boldsymbol{v} - (\bar{\boldsymbol{v}} \bullet \nabla)\boldsymbol{v} - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \qquad \cdots(24)$$

Since the term $-\lambda C$ in Eq.(7) is attenuation one, we let $\lambda = 0$. So, we have the following corresponding relation.

$$C \leftrightarrow \boldsymbol{v}, \quad k \leftrightarrow D, \quad Q \leftrightarrow -\frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho}$$

If we regard that the $\bar{v}$ is a constant vector during infinitesimal time, we can obtain the highly accurate approximate solution of NSE by applying the solution of ADE based on the aforementioned corresponding relation.

$$
\begin{aligned}
& \boldsymbol{v}(x,y,z,t) \\
& = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{v}_0(\xi,\eta,\zeta) G(x,y,z,t,\xi,\eta,\zeta,0) d\xi d\eta d\zeta \\
& + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^t \boldsymbol{Q}(\xi,\eta,\zeta,\tau) G(x,y,z,t,\xi,\eta,\zeta,\tau) d\tau \right\} d\xi d\eta d\zeta
\end{aligned}
$$

where $\quad \boldsymbol{Q} = -\frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho}$

$$\cdots(25)$$

This means the velocity (or momentum) at the point $\boldsymbol{r}_p = (x, y, z)$ is decided by taking the diffusion of the momentum from the surrounding points that can be computed by the convolution of the Green function (Fig.1).

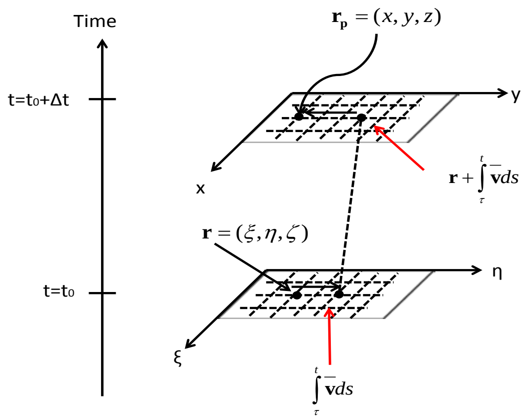

Fig.1. The situation of receiving the momentum (or velocity) diffusion from the surrounding points decided by the past time velocity (or momentum).

5. Checking the analytical approximate solution of the NSE

By substituting the Eq.(25) for $\boldsymbol{v}$ in Eq.(24), we can check whether the Eq.(25) is the analytical approximate solution of the NSE or not, in the same way as the check of the exact analytical solution of the ADE.
The LHS of Eq.(24) is shown as follows.

$$\frac{\partial v(x,y,z,t)}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{v}_0(\xi,\eta,\zeta)\cdot K \cdot \left[-\frac{3k}{\sigma_0^2} + \frac{k}{\sigma_0^4} \left\langle \mathbf{Z}_0 + \frac{\sigma_0^2}{k}\bar{\mathbf{v}}(t) \middle| \mathbf{Z}_0 \right\rangle \right] \exp\left(-\frac{\|\mathbf{Z}_0\|^2}{2\sigma_0^2} \right) d\xi d\eta d\zeta$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^t Q \cdot K \left[-\frac{3k}{\sigma^2} + \frac{k}{\sigma^4} \left\langle \mathbf{Z} + \frac{\sigma^2}{k}\bar{\mathbf{v}}(t) \middle| \mathbf{Z} \right\rangle \right] \exp\left(-\frac{\|\mathbf{Z}\|^2}{2\sigma^2} \right) d\tau \right\} d\xi d\eta d\zeta$$
$$-\frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho}$$

where $\bar{\boldsymbol{v}} = (u,v,w),\ \sigma = \sqrt{2k(t-\tau)}, K = \left(\sigma\sqrt{\pi}\right)^{-3}, \mathbf{Q} = -\frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho},$

$$\mathbf{Z} = \mathbf{r_p} - \left(\mathbf{r} + \int_\tau^t \bar{\mathbf{v}} ds \right) \quad \ldots(26)$$

And, we can show that the RHS of Eq.(24) becomes entirely the same as above LHS of Eq.(24).
Then, we consider that Eq.(25) is a highly accurate approximate solution of the NSE.

6. Conclusion

In this paper, we have focused on the similarity between the Navier-Stokes Equation (NSE) and Advection Diffusion Equation (ADE). Then, we have described the explicit similarity of them and pointed out the corresponding term between both equations.
Moreover, based on the viewpoint that NSE is a kind of advection (or convection) diffusion equation of momentum, we have derived an approximate solution of NSE by applying the exact analytical solution of the ADE to the NSE.
Although this solution can be regarded as an approximation, we consider that it is a highly accurate approximation that can be remarkably close to the exact analytical solution of NSE.

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