Numerical Analysis of A Mixed Finite Element Method for Rosenau-Burgers Equation

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Abstract. An H^1-Galerkin mixed finite element method (H^1MFEM) is proposed and analyzed for the fourth-order nonlinear Rosenau-Burgers equation. By introducing three auxiliary variables, the first-order system of four equations is formulated. The fully discrete scheme is studied for problem and optimal a priori error estimates for L^2 and H^1-norms for the scalar unknown, first derivative, second derivative and third derivative are obtained simultaneously.

Introduction

In this article, we consider the following fourth-order Rosenau-Burgers equations

$$\begin{cases} u_t + u_{xxxxt} - u_{xx} + u_x + uu_x = 0, (x,t) \in \Omega \times J, \\ u(0,t) = u(1,t) = 0, u_{xx}(0,t) = u_{xx}(1,t) = 0, t \in \overline{J}, \\ u(x,0) = u_0(x), x \in \Omega. \end{cases} \tag{1.1}$$

where $\Omega = (0,1), J = (0,T]$.

Equation (1) is called as usual Rosenau-Burgers equation arises in some natural phenomena, such as, in bore propagation and in water waves[1, 2, 3, 4, 5]. From the literature review, we can see that many numerical schemes are analyzed. However, there is a limited study for finite element methods of Rosenau-Burgers equation.

Pani [6] (in 1998) proposed the H^1MFEM of the linear parabolic equation. Compared to standard mixed methods, the proposed method has several attractive features. First, they do not satisfy the LBB consistency condition. Second, the polynomial degrees of the finite element spaces V_h and W_h may be different. Recently, many researchers have studied H^1MFEM for second-order partial differential equations [7,8,9,10,11,12]. In 2012, Liu et al. [12] first proposed and studied the H^1MFEMs for fourth-order linear parabolic equation ($u_t + ku_{xxxx} = f(x,t)$, $\mathbf{0} < k \in R$). However, the convergence of H^1MFEM for fourth-order nonlinear Rosenau-Burgers has not been studied in the literatures. In this paper, we will derive the fully discrete error analysis of the H^1MFEM for nonlinear Rosenau-Burgers equation.

Throughout this paper, $C > 0$ will be denoted as a generic constant which is free of the space-time mesh parameter h and Δt.

H^1-Galerkin mixed scheme and some lemmas

We introduce three auxiliary variables $q = u_x$, $v = q_x$, $\sigma = v_x$ and reformulate the Rosenau-Burgers equation (1.1) as the first-order system

$$q = u_x,\ v = q_x,\ \sigma = v_x,\ u_t + q + uq - v + \sigma_{xt} = 0 \tag{2.1}$$

Then a mixed weak formulation is to seek $\{u, v; q, \sigma\} : [0,T] \to H_0^1 \times H^1$ satisfying

$$\begin{cases}(a)\ (u_x,\chi_x)=(q,\chi_x),\forall\chi\in H_0^1,\\(b)\ (v_x,\omega_x)=(\sigma,\omega_x),\forall\omega\in H_0^1,\\(c)\ (\sigma,\psi)+(q_x,\psi_x)=0,\forall\psi\in H_0^1,\\(d)\ (q_t,\phi)-(\sigma_{xt},\phi_x)=(q+uq-v,\phi_x),\forall\phi\in H^1.\end{cases}\tag{2.2}$$

With finite dimensional spaces $(V_h,W_h)\subset(H_0^1,H^1)$, the mixed scheme for (2.2) is to seek $\{u_h,v_h;q_h,\sigma_h\}:[0,T]\to V_h\times W_h$ such that

$$\begin{cases}(a)\ (u_{hx},\chi_{hx})=(q_h,\chi_{hx}),\forall\chi_h\in V_h,\\(b)\ (v_{hx},\omega_{hx})=(\sigma_h,\omega_{hx}),\forall\omega_h\in V_h,\\(c)\ (\sigma_h,\psi_h)+(q_{hx},\psi_{hx})=0,\forall\psi_h\in W_h,\\(d)\ (q_{ht},\phi_h)-(\sigma_{hxt},\phi_{hx})=(q_h+u_hq_h-v_h,\phi_{hx}),\forall\phi_h\in W_h.\end{cases}\tag{2.3}$$

For use in the error analysis, we introduce two important projections [6, 13].

Lemma 1 The elliptic projection $\tilde{y}_h\in V_h$ is defined by

$$(y_x-\tilde{y}_{hx},g_{hx})=0,g_h\in V_h.\tag{2.4}$$

Then the following error estimates are obtained

$$\|y-\tilde{y}_h\|_j\le Ch^{k+1-j}\|y\|_{k+1},j=0,1.\tag{2.5}$$

Lemma 2 A Ritz projection $\tilde{z}_h\in W_h$ is defined by

$$A(z-\tilde{z}_h,w_h)=\mathbf{0},w_h\in W_h.\tag{2.6}$$

where $A(z,w)=(z_x,w_x)+\lambda(z,w).$ Here λ is chosen to satisfy $A(w,w)\ge\mu_0\|w\|_1^2,w\in H^1,0<\mu_0\in R.$ Then the following error estimates

$$\|z-\tilde{z}_h\|_j\le Ch^{r+1-j}\|z\|_{r+1},\|z_t-\tilde{z}_{ht}\|_j\le Ch^{r+1-j}\|z_t\|_{r+1},\ \ j=0,1\tag{2.7}$$

can be found.

A priori error estimates for fully discrete scheme

Let $0=t_0<t_1<t_2<\cdots<t_M=T$ be a given partition of the time interval $[0,T]$ with step length $t_n=n\Delta t,\Delta t=T/M$, for some positive integer M. For a smooth function ϕ on $[0,T]$, define $\phi^n=\phi(t_n)$ and $\bar{\partial}_t\phi^n=(\phi^n-\phi^{n-1})/\Delta t$.

Let U^n,Q^n,V^n and Z^n respectively, be the approximations of u,q,v and σ at $t=t_n$ which we shall define through the following scheme. Given $\{U^{n-1},V^{n-1};Q^{n-1},Z^{n-1}\}$ in $V_h\times W_h$,we now determine $\{U^n,V^n;Q^n,Z^n\}$ in $V_h\times W_h$ satisfying

$$\begin{cases}(a)\ (U_x^n,\chi_{hx})=(Q^n,\chi_{hx}),\forall\chi_h\in V_h,\\(b)\ (V_x^n,\omega_{hx})=(Z^n,\omega_{hx}),\forall\omega_h\in V_h,\\(c)\ (Z^n,\psi_h)+(Q_x^n,\psi_{hx})=0,\forall\psi_h\in W_h,\\(d)\ (\bar{\partial}_tQ^n,\phi_h)-(\bar{\partial}_tZ_x^n,\phi_{hx})=(Q^n+U^{n-1}Q^n-V^n,\phi_{hx}),\forall\phi_h\in W_h.\end{cases}\tag{3.1}$$

For fully discrete error estimates, we now split the errors

$$u(t_n)-U^n=(u(t_n)-\tilde{u}_h(t_n))+(\tilde{u}_h(t_n)-U^n)=\eta^n+\varsigma^n,\ v(t_n)-V^n=(v(t_n)-\tilde{v}_h(t_n))+(\tilde{v}_h(t_n)-V^n)=\tau^n+\theta^n,$$

$$q(t_n)-Q^n=(q(t_n)-\tilde{q}_h(t_n))+(\tilde{q}_h(t_n)-Q^n)=\rho^n+\xi^n,\ \sigma(t_n)-Z^n=(\sigma(t_n)-\tilde{\sigma}_h(t_n))+(\tilde{\sigma}_h(t_n)-Z^n)=\delta^n+\gamma^n.$$

Using (3) and (21), we then obtain

$$\begin{cases}(a)\ (\varsigma_x^n,\chi_{hx})=(\rho^n,\chi_{hx})+(\xi^n,\chi_{hx}),\forall\chi_h\in V_h,\\(b)\ (\theta_x^n,\omega_{hx})=(\delta^n,\omega_{hx})+(\gamma^n,\omega_{hx}),\forall\omega_h\in V_h,\\(c)\ (\gamma^n,\psi_h)+A(\xi^n,\psi_h)=-(\delta^n,\psi_h)+\lambda(\xi^n+\rho^n,\psi_h),\forall\psi_h\in W_h,\\(d)\ (\bar{\partial}_t\xi^n,\phi_h)-A(\bar{\partial}_t\gamma^n,\phi_h)=-(\bar{\partial}_t\rho^n+\varepsilon^n,\phi_h)-\lambda(\bar{\partial}_t\delta^n+\bar{\partial}_t\gamma^n,\phi_h)+(u^nq^n-U^{n-1}Q^n,\phi_{hx})\\\qquad+(u^nq^n-U^{n-1}Q^n,\phi_{hx})-(\tau^n+\theta^n,\phi_{hx})+(\rho^n+\xi^n,\phi_{hx}),\forall\phi_h\in W_h.\end{cases}\tag{3.2}$$

where $\varepsilon^n = q_t(t_n) - \bar{\partial}_t q(t_n)$.

Theorem 1 Assume that $Z^0 = \tilde{q}_h(0)$ with $q_0 = u_{x0}$. Then there exists a constant $C > 0$ free of h and Δt such that for $0 < \Delta t < \Delta t_0$ and $j = 0,1$

$$\left\| u^J - U^J \right\|_j + \left\| v^J - V^J \right\|_j \le C\left(h^{\min(k+1-j,r+1)} + \Delta t\right), \left\| q^J - Q^J \right\|_j + \left\| \sigma^J - Z^J \right\|_j \le C\left(h^{\min(k+1,r+1-j)} + \Delta t\right),$$

Proof. Choosing $\chi_h = \varsigma^n, \omega_h = \theta^n$ in (3.2a) and (3.2b), respectively, we use Cauchy-Schwarz inequality, Young inequality and Poincare inequality to have

$$\left\|\varsigma^n\right\|^2 \le C\left\|\varsigma_x^n\right\|^2 \le C\left(\left\|\rho^n\right\|^2 + \left\|\xi^n\right\|^2\right), \quad \left\|\theta^n\right\|^2 \le C\left\|\theta_x^n\right\|^2 \le C\left(\left\|\delta^n\right\|^2 + \left\|\gamma^n\right\|^2\right). \tag{3.3}$$

Choose $\psi_h = \bar{\partial}_t \gamma^n$ and $\phi_h = \xi^n$ in (3.2c) and (3.2d), respectively, to obtain

$$\begin{cases} (a)\ \frac{1}{2}\bar{\partial}_t\left\|\gamma^n\right\|^2 + \left(\xi_x^n, \bar{\partial}_t\gamma_x^n\right) \le -\left(\delta^n, \bar{\partial}_t\gamma^n\right) + \lambda\left(\rho^n, \bar{\partial}_t\gamma^n\right), \\ (b)\ \frac{1}{2}\bar{\partial}_t\left\|\xi^n\right\|^2 - \left(\bar{\partial}_t\gamma_x^n, \xi_x^n\right) \le -\left(\bar{\partial}_t\rho^n + \varepsilon^n, \xi^n\right) - \lambda\left(\bar{\partial}_t\delta^n, \xi^n\right) + \left(u^n q^n - U^{n-1}Q^n, \xi_x^n\right) - \left(\tau^n + \theta^n, \xi_x^n\right) + \left(\rho^n + \xi^n, \xi_x^n\right). \end{cases} \tag{3.4}$$

Note that

$$\left(\delta^n, \bar{\partial}_t\gamma^n\right) = \frac{\left(\gamma^n, \delta^n\right) - \left(\gamma^{n-1}, \delta^{n-1}\right)}{\Delta t} - \left(\bar{\partial}_t\delta^n, \gamma^{n-1}\right), \qquad \lambda\left(\rho^n, \bar{\partial}_t\gamma^n\right) = \lambda\frac{\left(\gamma^n, \rho^n\right) - \left(\gamma^{n-1}, \rho^{n-1}\right)}{\Delta t} - \lambda\left(\bar{\partial}_t\rho^n, \gamma^{n-1}\right). \tag{3.5}$$

Substitute (3.5) into (3.4b) to get

$$\frac{1}{2}\bar{\partial}_t\left\|\gamma^n\right\|^2 + \left(\xi_x^n, \bar{\partial}_t\gamma_x^n\right) \le -\frac{\left(\gamma^n, \delta^n\right) - \left(\gamma^{n-1}, \delta^{n-1}\right)}{\Delta t} + \left(\bar{\partial}_t\delta^n, \gamma^{n-1}\right) + \lambda\frac{\left(\gamma^n, \rho^n\right) - \left(\gamma^{n-1}, \rho^{n-1}\right)}{\Delta t} - \lambda\left(\bar{\partial}_t\rho^n, \gamma^{n-1}\right). \tag{3.6}$$

Adding (3.6) and (3.4b) and using Cauchy-Schwarz inequality and Young inequality, we can get

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t\left\|\xi^n\right\|^2 + \frac{1}{2}\bar{\partial}_t\left\|\gamma^n\right\|^2 &= -\frac{\left(\gamma^n, \delta^n\right) - \left(\gamma^{n-1}, \delta^{n-1}\right)}{\Delta t} + \left(\bar{\partial}_t\delta^n, \gamma^{n-1}\right) - \lambda\left(\bar{\partial}_t\rho^n, \gamma^{n-1}\right) + \lambda\frac{\left(\gamma^n, \rho^n\right) - \left(\gamma^{n-1}, \rho^{n-1}\right)}{\Delta t} \\ &\quad - \left(\bar{\partial}_t\rho^n + \varepsilon^n, \xi^n\right) - \lambda\left(\bar{\partial}_t\delta^n, \xi^n\right) + \left(u^n q^n - U^{n-1}Q^n, \xi_x^n\right) - \left(\tau^n + \theta^n, \xi_x^n\right) + \left(\rho^n + \xi^n, \xi_x^n\right) \\ &\le -\frac{\left(\gamma^n, \delta^n\right) - \left(\gamma^{n-1}, \delta^{n-1}\right)}{\Delta t} + \lambda\frac{\left(\gamma^n, \rho^n\right) - \left(\gamma^{n-1}, \rho^{n-1}\right)}{\Delta t} \\ &\quad + C(\left\|\rho^n\right\|^2 + \left\|\delta^n\right\|^2 + \left\|\bar{\partial}_t\rho^n\right\|^2 + \left\|\bar{\partial}_t\delta^n\right\|^2 + \left\|\tau^n\right\|^2 + \left\|\xi^n\right\|_1^2 + \left\|\varepsilon^n\right\|^2 + \left\|\theta^n\right\|^2 + \left\|\gamma^{n-1}\right\|^2) + \left|\left(u^n q^n - U^{n-1}Q^n, \xi_x^n\right)\right|. \end{aligned} \tag{3.7}$$

Noting that

$$\begin{aligned} \left|\left(u^n q^n - U^{n-1}Q^n, \phi_{hx}\right)\right| &= \left|\left(\left(u^n q^n - u^{n-1}q^n\right) + \left(u^{n-1}q^n - U^{n-1}q^n\right) + \left(U^{n-1}q^n - U^{n-1}Q^n\right), \phi_{hx}\right)\right| \\ &\le \left\|q^n\right\|_{0,\infty}\left\|u^n - u^{n-1}\right\|\left\|\phi_{hx}\right\| + \left\|q^n\right\|_{0,\infty}\left\|u^{n-1} - U^{n-1}\right\|\left\|\phi_{hx}\right\| + \left\|U^{n-1}\right\|_{0,\infty}\left\|q^n - Q^n\right\|\left\|\phi_{hx}\right\| \\ &\le C\left[\Delta t\int_{t_{n-1}}^{t_n}\left\|u_t\right\|^2 ds + \left\|\eta^{n-1}\right\|^2 + \left\|\varsigma^{n-1}\right\|^2\left\|\rho^n\right\|^2 + \left\|\xi^n\right\|^2\right] + C\left\|\phi_{hx}\right\|^2. \end{aligned} \tag{3.8}$$

Choose $\phi_h = \xi^n$ in (3.8) to obtain

$$\left|\left(u^n q^n - U^{n-1}Q^n, \xi_x^n\right)\right| \le C\left[\Delta t\int_{t_{n-1}}^{t_n}\left\|u_t\right\|^2 ds + \left\|\eta^{n-1}\right\|^2 + \left\|\varsigma^{n-1}\right\|^2 + \left\|\rho^n\right\|^2 + \left\|\xi^n\right\|^2\right] + C\left\|\xi_x^n\right\|^2. \tag{3.9}$$

Substituting (3.9) into (3.8) and using (3.3), we obtain

$$\begin{aligned} \frac{1}{2}\bar{\partial}_t\left\|\xi^n\right\|^2 + \frac{1}{2}\bar{\partial}_t\left\|\gamma^n\right\|^2 &\le -\frac{\left(\gamma^n, \delta^n\right) - \left(\gamma^{n-1}, \delta^{n-1}\right)}{\Delta t} + \lambda\frac{\left(\gamma^n, \rho^n\right) - \left(\gamma^{n-1}, \rho^{n-1}\right)}{\Delta t} \\ &\quad + C(\Delta t\int_{t_{n-1}}^{t_n}\left\|u_t\right\|^2 ds + \left\|\rho^{n-1}\right\|^2 + \left\|\xi^{n-1}\right\|^2 + \left\|\eta^{n-1}\right\|^2 + \left\|\rho^n\right\|^2 \\ &\quad + \left\|\delta^n\right\|^2 + \left\|\bar{\partial}_t\rho^n\right\|^2 + \left\|\bar{\partial}_t\delta^n\right\|^2 + \left\|\varepsilon^n\right\|^2 + \left\|\gamma^{n-1}\right\|^2 + \left\|\tau^n\right\|^2 + \left\|\xi^n\right\|_1^2). \end{aligned} \tag{3.10}$$

Sum (3.10) from $n = 1$ to J to get

$$\left\|\xi^J\right\|^2+\left\|\gamma^J\right\|^2 \le C\left(\left\|\rho^J\right\|^2+\left\|\delta^J\right\|^2\right)+C\Delta t\sum_{n=1}^{J}(\left\|\rho^n\right\|^2+\left\|\eta^n\right\|^2+\left\|\delta^n\right\|^2+\left\|\bar{\partial}_t\rho^n\right\|^2$$
$$+\left\|\bar{\partial}_t\delta^n\right\|^2+\left\|\varepsilon^n\right\|^2+\left\|\tau^n\right\|^2+\left\|\xi^n\right\|_1^2+\left\|\gamma^{n-1}\right\|^2)+C(\Delta t)^2\int_0^{t_J}\left\|u_t\right\|^2dt. \quad (3.11)$$

Choose $\psi_h=\bar{\partial}_t\xi^n$ and $\phi_h=\gamma^n$ in (3.1c) and (3.1d), respectively, to obtain

$$\begin{cases} (a)\ \left(\gamma^n,\bar{\partial}_t\xi^n\right)+\dfrac{1}{2}\bar{\partial}_t\left\|\xi_x^n\right\|^2 \le -\left(\delta^n,\bar{\partial}_t\xi^n\right)+\lambda\left(\rho^n,\bar{\partial}_t\xi^n\right), \\ (b)\ -\left(\bar{\partial}_t\xi^n,\gamma^n\right)+\dfrac{1}{2}\bar{\partial}_t\left\|\gamma_x^n\right\|^2 \le \left(\bar{\partial}_t\rho^n+\varepsilon^n,\gamma^n\right)+\lambda\left(\bar{\partial}_t\delta^n,\gamma^n\right) \\ \qquad +\left(u^nq^n-U^{n-1}Q^n,\gamma_x^n\right)+\left(\tau^n+\theta^n,\gamma_x^n\right)-\left(\rho^n+\xi^n,\gamma_x^n\right). \end{cases} \quad (3.12)$$

Note that

$$\left(\delta^n,\bar{\partial}_t\xi^n\right)=\frac{\left(\xi^n,\delta^n\right)-\left(\xi^{n-1},\delta^{n-1}\right)}{\Delta t}-\left(\bar{\partial}_t\delta^n,\xi^{n-1}\right),\ \lambda\left(\rho^n,\bar{\partial}_t\xi^n\right)=\lambda\frac{\left(\xi^n,\rho^n\right)-\left(\xi^{n-1},\rho^{n-1}\right)}{\Delta t}-\lambda\left(\bar{\partial}_t\rho^n,\xi^{n-1}\right). \quad (3.13)$$

Substitute (3.13) into (3.12b) to get

$$\left(\gamma^n,\bar{\partial}_t\xi^n\right)+\frac{1}{2}\bar{\partial}_t\left\|\xi_x^n\right\|^2$$
$$\le -\frac{\left(\xi^n,\delta^n\right)-\left(\xi^{n-1},\delta^{n-1}\right)}{\Delta t}+\left(\bar{\partial}_t\delta^n,\xi^{n-1}\right)+\lambda\frac{\left(\xi^n,\rho^n\right)-\left(\xi^{n-1},\rho^{n-1}\right)}{\Delta t}-\lambda\left(\bar{\partial}_t\rho^n,\xi^{n-1}\right). \quad (3.14)$$

Adding (3.12b) and (3.14), we get

$$\frac{1}{2}\bar{\partial}_t\left\|\xi_x^n\right\|^2+\frac{1}{2}\bar{\partial}_t\left\|\gamma_x^n\right\|^2 \le -\frac{\left(\xi^n,\delta^n\right)-\left(\xi^{n-1},\delta^{n-1}\right)}{\Delta t}+\left(\bar{\partial}_t\delta^n,\xi^{n-1}\right)+\lambda\frac{\left(\xi^n,\rho^n\right)-\left(\xi^{n-1},\rho^{n-1}\right)}{\Delta t}-\lambda\left(\bar{\partial}_t\rho^n,\xi^{n-1}\right)$$
$$+\left(\bar{\partial}_t\rho^n+\varepsilon^n,\gamma^n\right)+\lambda\left(\bar{\partial}_t\delta^n,\gamma^n\right)+\left(\tau^n+\theta^n,\gamma_x^n\right)+\left(u^nq^n-U^{n-1}Q^n,\gamma_x^n\right)-\left(\rho^n+\xi^n,\gamma_x^n\right). \quad (3.15)$$

Choose $\phi_h=\gamma^n$ in (3.8) to obtain

$$\left|\left(u^nq^n-U^{n-1}Q^n,\gamma_x^n\right)\right| \le C[\,\Delta t\int_{t_{n-1}}^{t_n}\left\|u_t\right\|^2ds+\left\|\eta^{n-1}\right\|^2+\left\|\varsigma^{n-1}\right\|^2+\left\|\rho^n\right\|^2+\left\|\xi^n\right\|^2\,]+\frac{1}{4}\left\|\gamma_x^n\right\|^2. \quad (3.16)$$

Substitute (3.16) into (3.15) and apply Cauchy-Schwarz inequality and Young inequality to get

$$\frac{1}{2}\bar{\partial}_t\left\|\xi_x^n\right\|^2+\frac{1}{2}\bar{\partial}_t\left\|\gamma_x^n\right\|^2 \le -\frac{\left(\xi^n,\delta^n\right)-\left(\xi^{n-1},\delta^{n-1}\right)}{\Delta t}+\lambda\frac{\left(\xi^n,\rho^n\right)-\left(\xi^{n-1},\rho^{n-1}\right)}{\Delta t}$$
$$+C[\,\Delta t\int_{t_{n-1}}^{t_n}\left\|u_t\right\|^2ds+\left\|\eta^{n-1}\right\|^2+\left\|\varsigma^{n-1}\right\|^2+\left\|\rho^n\right\|^2+\left\|\xi^{n-1}\right\|^2+\left\|\xi^n\right\|^2+\left\|\tau^n\right\|^2$$
$$+\left\|\theta^n\right\|^2+\left\|\delta^n\right\|^2+\left\|\gamma^n\right\|^2+\left\|\bar{\partial}_t\delta^n\right\|^2+\left\|\bar{\partial}_t\rho^n\right\|^2+\left\|\varepsilon^n\right\|^2\,]+\frac{1}{2}\left\|\gamma_x^n\right\|^2. \quad (3.17)$$

Sum (3.17) from $n=1$ to J and use Cauchy-Schwarz inequality, Young inequality and (3.12)-(3.13) to get

$$\left\|\xi_x^J\right\|^2+\left\|\gamma_x^J\right\|^2 \le C\left(\left\|\xi^J\right\|^2+\left\|\rho^J\right\|^2+\left\|\delta^J\right\|^2\right)+C(\Delta t)^2\int_0^{t_J}\left\|u_t\right\|^2dt+C\Delta t\sum_{n=1}^{J}[\left\|\eta^{n-1}\right\|^2$$
$$+\left\|\rho^n\right\|^2+\left\|\xi^n\right\|^2+\left\|\tau^n\right\|^2+\left\|\delta^n\right\|^2+\left\|\gamma^n\right\|_1^2+\left\|\bar{\partial}_t\delta^n\right\|^2+\left\|\bar{\partial}_t\rho^n\right\|^2+\left\|\varepsilon^n\right\|^2\,]. \quad (3.18)$$

Combining (3.11) and (3.12), we can obtain

$$\left\|\xi^J\right\|_1^2+\left\|\gamma^J\right\|_1^2 \le C\left(\left\|\rho^J\right\|^2+\left\|\delta^J\right\|^2\right)+C(\Delta t)^2\int_0^{t_J}\left\|u_t\right\|^2dt$$
$$+C\Delta t\sum_{n=1}^{J}(\left\|\eta^n\right\|^2+\left\|\rho^n\right\|^2+\left\|\xi^n\right\|_1^2+\left\|\gamma^n\right\|_1^2+\left\|\tau^n\right\|^2+\left\|\delta^n\right\|^2+\left\|\bar{\partial}_t\delta^n\right\|^2+\left\|\bar{\partial}_t\rho^n\right\|^2+\left\|\varepsilon^n\right\|^2\,]. \quad (3.19)$$

Using discrete Gronwall lemma, we obtain

$$\left\|\xi^J\right\|_1^2+\left\|\gamma^J\right\|_1^2 \le C\left(\left\|\rho^J\right\|^2+\left\|\delta^J\right\|^2\right)+C(\Delta t)^2\int_0^{t_J}\left\|u_t\right\|^2dt$$
$$+C\Delta t\sum_{n=1}^{J}(\left\|\eta^n\right\|^2+\left\|\rho^n\right\|^2+\left\|\tau^n\right\|^2+\left\|\delta^n\right\|^2+\left\|\bar{\partial}_t\delta^n\right\|^2+\left\|\bar{\partial}_t\rho^n\right\|^2+\left\|\varepsilon^n\right\|^2\,]. \quad (3.20)$$

Noting that

$$\left\|\overline{\partial}_t\rho^n\right\|^2 \le Ch^{2(r+1)}\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left\|q_t(s)\right\|_{r+1}^2 ds,\ \left\|\overline{\partial}_t\delta^n\right\|^2 \le Ch^{2(r+1)}\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left\|\sigma_t(s)\right\|_{r+1}^2 ds,\ \left\|\varepsilon^n\right\|^2 \le C\Delta t\int_{t_{n-1}}^{t_n}\left\|q_{tt}(s)\right\|^2 ds.$$

Substitute the above inequalities into (3.20) to obtain

$$\begin{aligned}\left\|\xi^J\right\|_1^2+\left\|\gamma^J\right\|_1^2 &\le C\left(\left\|\rho^J\right\|^2+\left\|\delta^J\right\|^2\right)+C(\Delta t)^2\int_0^{t_J}(\left\|u_t(s)\right\|^2+\left\|q_{tt}(s)\right\|^2)dt\\ &+C\Delta t\sum_{n=1}^{J}[\left\|\eta^n\right\|^2+\left\|\rho^n\right\|^2+\left\|\tau^n\right\|^2+\left\|\delta^n\right\|^2]+h^{2(r+1)}\int_0^{t_J}(\left\|q_t(s)\right\|_{r+1}^2+\left\|\sigma_t(s)\right\|_{r+1}^2)ds).\end{aligned} \tag{3.21}$$

Combining (3.21), (3.3), (2.4)-(2.7) with the triangle inequality, we obtain the conclusion of Theorem 1.

Concluding remarks

Compared to the study of the H^1MFEM for second-order partial differential equations, the fourth-order nonlinear problems have not been studied in the literatures. In this paper, we study the H^1MFEM for solving nonlinear Rosenau-Burgers equation with fourth-order spatial derivative. We obtain the optimal a priori error estimates in L^2-and H^1-norm for four variables. Compared to other mixed methods, our methods can approximate the scalar unknown u, the gradient $q=u_x$, second-order derivative term $v=q_x=u_{xx}$ and third-order derivative term $\sigma=v_x=u_{xxx}$, simultaneously.

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