A New Theorem on Bargaining Sets in TU Games

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Keywords: Soccer Robot; Mechanical Analysis; Optimal Design

Abstract. In this paper, we provide a new existence theorem by proving that Mas-Colell bargaining sets exist for all TU games.

Introduction

Let $N = \{1, 2, ..., n\}$ be the set of n players. Any subset of N is called a coalition.

Definition 1.1. ***A*** *cooperative game* (or a TU game) in characteristic function form with player set N is a map $\upsilon : 2^{\mathrm{N}} \to \mathbb{R}$ with the property $\upsilon(\phi) = 0$.

A payoff vector $x \in \mathbb{R}^n$ is said to be individual rational if $x_i \ge \upsilon(\{i\})$ for each $i \in N$.

Definition 1.2. The *imputation set* $I(\upsilon)$ of a cooperative game υ is the set

$$I(\upsilon) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i \in N} x_i = \upsilon(N), x_i \ge \upsilon(\{i\}) \text{ for each } i \in N \right\}$$

Cooperative games have been studied extensively in the literature. A central question in cooperative games is to study solution concepts and their relationships, those well-known solution concepts include cores, stable sets, Shapley values, bargaining sets, and so on.

To state Vohra's result formally, let us recall some necessary concepts from [4].

A *non-transferable utility game* (NTU game) in characteristic function form is defined as a pair (N, V), where $V: 2^N \to \mathbb{R}^N$ is a correspondence satisfying

(i) for all non-empty $S \in 2^N$, $V(S)$ is non-empty, closed, and comprehensive,

(ii) for all $i \in N$, $V(\{i\}) = \{x \in \mathbb{R}^N \,|\, x_i \le 0\}$,

(iii) for all $S \in 2^N$, $V(S)_S \cap \mathbb{R}^S_+$ is bounded.

A TU game v in characteristic function form is equivalent to an NTU game (N, V) such that for every non-empty $S \in 2^N$,

$$V(S) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in S} x_i \le \upsilon(S) \right\}. \qquad (1.1)$$

In fact, Condition (ii) in the definition above by Vohra also requires $\upsilon(\{i\}) = 0$ for all $i \in N$, which can be achieved by zero normalization.

Weak Superadditivity (*version* 1): For any $S \in 2^N$ and $i \notin S$, if $x \in V(S)$, then $y \in V(S \cup \{i\})$, where $y_i = 0$ and $y_j = x_j$ for $j \ne i$.

This has the following equivalent form given in [2].

Weak *Superadditivity* (*version 2*): An NTU game (N, V) is weakly superadditive if for every $i \in N$ and every $S \subseteq N \setminus \{i\}$ satisfying $S \ne \phi$, $V(S) \times V(\{i\}) \subseteq V(S \cup \{i\})$.

Clearly, for TU games, the weak superadditivity is equivalent to the following according to version 2 and (1.1).

Weak *Superadditivity for TU games*: $\upsilon(S)+\upsilon(\{i\})\le\upsilon(S\cup\{i\})$ for each $S\subseteq N$ and each $i\in N\setminus S$.

In 1991, Vohra [4] proved the following existence theorem for Mas-Colell bargaining sets.

Theorem 1.3 (Vohra, 1991). If υ is a weakly superadditive TU game, then the Mas-Colell bargaining set $MB(\upsilon)$ of υ is non-empty.

In this paper, we prove the following stronger existence theorem for Mas-Colell bargaining sets in TU games.

Theorem 1.4. If υ is a TU game such that $\upsilon(S)\le\upsilon(N)$ for each $S\subseteq N$, then the Mas-Colell bargaining set $MB(\upsilon)$ of υ is non-empty.

Lemma 1.7. Let υ be a TU game and let υ_0 be the zero-normalized game of υ. Then $x\in MB(\upsilon)$ if and only if $x'\in MB(\upsilon_0)$, where $x_i'=x_i-\upsilon(i)$ for each $i\in N$.

Proof of Theorem 1.4

In this section, we will give a proof for Theorem 1.4 by proving the following Theorem 2.2 which implies Theorem 1.4. Our proof is motivated in part by the ideas from [4] and [5]. Let υ be a TU game. For an imputation $x\in I(\upsilon)$ and a coalition $S\subseteq N$, the excess of S at x is

$$e(S,x)=\upsilon(S)-\sum_{k\in S}x_k$$

Clearly, we have following remark from the definitions.

Remark 2.1. An objection (S,y) at x exists if and only if $e(S,x)>0$.

Next, for the purpose of overcoming difficulties in our proof for Theorem 1.4, we in- troduce strong counterobjection as follows, where the special conditions imposed on strong counterobjection is just a technical device.

Strong Counterobjection: Given an objection (S,y) at $x\in I(\upsilon)$, a strong *counterobjection* to (S,y) at x is a pair (T,z), where *T* is a coalition such that $T\setminus S\ne\phi$ and there exists $h\in S\setminus T$ satisfying $y_h-x_h=\max\{y_i-x_i\,|\,i\in S\}>0$, and z is a vector in R^T satisfying that $z(T)=\sum_{i\in T}z_i=\upsilon(T), z_i\ge y_i$ for each $i\in S\cap T$, and $z_i\ge x_i+\dfrac{\sum_{j\in S\setminus T}(y_j-x_j)}{|T\setminus S|}$ for each $i\in T\setminus S$.

An imputation $x\in I(\upsilon)$ is said to belong to *strong Mas-Colell bargaining set* $MB_S(\upsilon)$ if for any objection (*S*, *y*) at x, there exists a strong counterobjection to it at x.

Theorem 2.2. If υ is a TU game such that $\upsilon(S)\le\upsilon(N)$ for each $S\subseteq N$, then the strong Mas-Colell bargaining set $MB_S(\upsilon)$ of υ is non-empty.

Lemma 2.3. Given an objection (*S*, *y*) at x and a non-empty coalition *T* such that $T\setminus S\ne\phi$ and there exists $h\in S\setminus\mathrm{T}$ satisfying $y_h-y_h=\max\{y_i-x_i\,|\,i\in S\}>0$, then a strong counterobjection (*T*, *z*) to (*S*, *y*) at x exists if and only if $e(T,x)\ge e(S,x)$.

Next we introduce the concept of balanced collection and a result from [11] which is needed in our proof.

Let Δ^N be the standard simplex:

$$\Delta^N=\left\{x\in R^N\,\middle|\,x_i\ge 0 \text{ for each } i\in N \textit{ and } \sum_{i=1}^{n}x_i=1\right\}.$$

Its *i*-th face is $\Delta^{N\setminus\{i\}}=\{x\in\Delta^N\,|\,x_i=0\}$. For each $S\subseteq N$, denote e^S the n-dimensional vector with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$.

Definition 2.4. A collection *B* of non-empty subsets (coalitions) of *N* is balanced if there exist positive numbers λ_S for $S \in B$ such that

$$\sum_{S \in B} \lambda_s e^S = e^N. \tag{2.1}$$

The numbers λ_S are called balancing coef f icients.

Clearly, the condition in (2.1) for a balanced collection *B* is equivalent to the following.

$$\sum_{S \in B : i \in S} \lambda s = 1 \text{ for each } i \in N. \tag{2.2}$$

The next theorem is proved by Zhou.

Theorem 2.5 (Zhou, 1994). If $\{O_S\}_{S \in N}$ is a family of open sets of Δ^N that satisfy

(1) $\Delta^{N \setminus \{i\}} \subseteq O_{\{i\}}$ for each $i \in N$ and

(2) $\bigcup_{S \in N} O_S = \Delta^N$,

then there is a balanced collection *B* of non-empty subsets (coalitions) of *N* such that $\bigcap_{S \in B} O_S \neq \phi$.

Let υ be a TU game. Note that the core $C(\upsilon)$ of υ consists of all $x \in I(\upsilon)$ such that $e(S,x) \leq 0$ for all $S \subseteq N$. It follows from Remark 2.1 that the core $C(\upsilon)$ is a subset of Mas-Colell bargaining set $MB(\upsilon)$. Thus, whenever υ has a non-empty core, $MB(\upsilon)$ is non-empty. This means that, when we deal with the existence of $MB(\upsilon)$, we may assume that $C(\upsilon) = \phi$, that is, for any $x \in I(\upsilon)$, there exists $S \subseteq N$ such that $e(S,x) > 0$. For each $x \in I(\upsilon)$, let $e_x = min\{e(S,x) \mid S \subseteq N \text{ with } e(S,x) > 0\}$ and set

$$\varepsilon_x = min\left\{\frac{1}{n} e_x, \frac{1}{n} \upsilon(N)\right\} \tag{2.3}$$

Then, under the assumption that $\upsilon(N) > 0$ and $C(\upsilon) = \phi$, $\varepsilon_x > 0$ for each $x \in I(\upsilon)$.

Let υ be a TU game and $x \in I(\upsilon)$. We say an objection (*S*, *y*) at x is *strongly justif -ied* if there is no strong counterobjection to (*S*, *y*) at x. For each non-empty $S \subseteq N$,define O_S as follows:

$$O_{\{i\}} = \{x \in I(\upsilon) \mid x_i < \varepsilon_x\} \text{ for each } i \in N,$$

$$O_S = \{x \in I(\upsilon) \mid \text{there exists a strongly justified objection } (S,y) \text{ } at \text{ } x\} \text{ if } |S| \geq 2.$$

The following fact follows from the definition immediately.

Fact 2.6. Let υ be a TU game with empty core and $\upsilon(N) > 0$. For each $i \in N$, $\Delta^{N \setminus \{i\}} \subseteq O_{\{i\}}$.

Lemma 2.7. Let υ be a TU game with $\upsilon(N) > 0$. Then, for each non-empty $S \subseteq N$, O_S is open.

Lemma 2.8. Let *v* be a TU game such that $v(N) > 0$ and $v(S) \leq v(N)$ for each $S \subseteq N$. Then for any balanced collection *B* of coalitions, $\bigcap_{S \in B} O_s = 0$.

The next lemma allows us to assume v(N) > 0 when dealing with the non-emptiness of strong Mas-Colell bargaining sets.

Lemma 2.9. Let *v* be a TU game and let *b* > 0 be such that v(N) + *b* > 0. Define v' to be the game such that $v'(S) = v(S) + \frac{|S|}{n} b$ for each $S \subseteq N$. Then $x \in MB_S(v)$ if and only if

$x^{'} \in MB_s(v^{'})$ where $x_i^{'} = x_i + \frac{b}{n}$ for each $i \in N$.

We now prove Theorem 1.4 by proving Theorem 2.2.

Proof of Theorem 2.2. Let v be a TU game such that v(S) $\leq$ v(N) for all S $\subseteq$ N. In view of Lemma 2.9, we may assume $v(N)$ > 0. In fact, if v(N) $\leq$ 0, then let b > 0 be such that v(N) + b > 0 and define $v^{'}$ to be the game such that $v^{'}(S) = v(S) + \frac{|s|}{n}b$ for each S $\subseteq$ N. Then v'(N) = v(N) + b > 0 and v'(S) $\leq$ v'(N) for each S $\subseteq$ N. By Lemma 2.9, $MB_s(v)$ is non-empty if and only if $MB_s(v^{'})$ is non-empty. Thus, we may assume v(N) > 0. If the core C(v) of v is non-empty, then we have the strong Mas-Colell bargaining set $MB_s(v)$ is non-empty. Thus, we may assume that the core C(v) is empty.

Recall that for each x $\in$ I(v), $\sum_{i=1}^{n} x_i = v(N) > 0$. We map Q = I(v) onto the standard simplex Δ^N by f:

$$f : x \to \frac{x}{\sum_{i=1}^{n} x_i}$$

Suppose, to the contrary, that the strong Mas-Colell bargaining set $MB_s(v)$ is empty. Then we have $Q \setminus \mathrm{U}_{0 \neq S \subseteq N} O_S = 0$. This means that Δ^N = f (Q) = $\mathrm{U}_{0 \neq S \subseteq N} f(O_S)$. By Fact 2.6, $\square^{N \setminus \{i\}} \subseteq f(O_{\{i\}})$ for each i $\in$ N. It follows from Theorem 2.5 that there is a balanced collection B of coalitions such that $\bigcap_{S \in B} f(\mathrm{O}_s) \neq 0$. But, by Lemma 2.8, we have

$\bigcap_{S \in B} O_s = \phi$ .It follows that $\bigcap_{S \in B} f(O_s) = \phi$,a contradiction.thus,the theorem holds.

Conclusion

In this paper, we proofed a stronger existence theorem by proving that Mas-Colell bargaining sets exist for all TU games.

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