Geometrically Distinct Solutions for a class of quasilinear Schrödinger equation with critical growth

Xiumei He[1, a], Xingping Li[2,b]

[1] Department of Mathematics and Statistics, Yunnan University, Kunming,650091,China

[1] Department of Mathematics, Kunming University, Kunming, Yunnan, 650214, China

[2] Department of Mathematics and Statistics, Yunnan University, Kunming,650091,China

[a]email:hexiumei2004@163.com, [b]email: lxp551@aliyun.com

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Abstract. In this paper, we consider the quasilinear Schrödinger equation with critical growth $-\Delta u - \lambda u \Delta u^2 + V(x)u = |u|^{2 \cdot 2^*-2} u + g(x,u), \quad x \in \square^N$ .We employ the perturbation approach developed by Xiangqing Liu [9] and the generalized linking approach and obtain the infinitely many geometrically distinct solutions.

Introduction and Preliminaries

In this paper, we study the following quasilinear Schrödinger equation with critical growth

$$-\Delta u - \lambda u \Delta u^2 + V(x)u = |u|^{2 \cdot 2^*-2} u + g(x,u), \quad x \in \square^N , \tag{1.1}$$

where $\lambda \geq 0, 2^* = \dfrac{2N}{N-2}(N \geq 3), g \in C(\square^N \times \square, \square)$ and $V \in C(\square^N, \square)$, whose weak variational formulation is to look for $u \in H^1(\square^N) \bigcap L^\infty(\square^N)$ such that

$$\int_{\square^N} \nabla u \nabla \varphi dx + 2\lambda \int_{\square^N} (u^2 \nabla u \nabla \varphi + |\nabla u|^2 u\varphi)dx + \int_{\square^N} V(x)u\varphi dx - \int_{\square^N} |u|^{2 \cdot 2^*-2} u\varphi dx - \int_{\square^N} g(x,u)\varphi dx = 0$$

for all $\varphi \in H^1(\square^N) \bigcap L^\infty(\square^N)$.

When $\lambda = 1$, the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(x,u), \quad x \in \square^N , \tag{1.2}$$

has been studied recently by several authors, see [2,3,6] and the refferences therein. Solutions of equation (1.2) are standing waves the following quasilinear Schrödinger equation of the form

$$i\psi_t + \Delta \psi - V(x)\psi + k\Delta(\alpha(|\psi|^2))\alpha'(|\psi|^2)\psi + g(x,\psi) = 0, \quad x \in \square^N , \tag{1.3}$$

where $V(x)$ is a given potential, k is a real constant, α and g are real functions. The quasilinear Schrödinger equations (1.3) are derived as models of several physical phenomena, such as see[8, 11]. It begins with [10] for the studies on mathematics. Several methods can be used to solve the equation (1.2), such as, the existence of a positive ground state solution has been studied in [14] by using a constrained minimization argument; the problem is transformed to a semilinear one in [1, 2] by a change of variables; Nehari method is used to get the existence results of ground state solutions in [12]. In [4], by a dual approach (precisely, a change of variables), an existence theorem for infinitely many periodic orbits of solutions for the equation (1.2) was obtained.

In this paper, our aim is to search the existence of infinitely many pairs of geometrically distinct solutions for problem (1.1) via the perturbation approach due to [9].

We need the following several notations. Let $H^1(\square^N) := \left\{ u \in L^2(\square^N) : \nabla u \in L^2(\square^N) \right\}$ with the inner product $\langle u,v \rangle_{H^1} = \int_{\square^N} (\nabla u \nabla v + uv)dx$ and the norm $\|u\|_{H^1}^2 = \int_{\square^N} (|\nabla u|^2 + |u|^2)dx$. Let the following

assumption (V) hold:

(V) $V \in C(\square^N, \square)$, $0 < V_0 := \inf_{x \in \square^N} V(x)$ and $V(x_1, \cdots, x_N)$ is 1-periodic in $x_i, i = 1, \cdots, N$.

Set $E = H^1(\square^N) \bigcap W^{1,4}(\square^N)$ with the norm $\|u\|_E = \|u\|_{H^1} + \|u\|_{W^{1,4}(\square^N)}$. Then E is a separable reflexive Banach space. The following embeddings are continuous: $E \to L^s(\square^N)$, $s \in [2, 22^*]$, where we denote by $\|\cdot\|_s$ the norm of $L^s(\square^N)$. The following embeddings are compact: $E \to L^s(\square^N)$, $s \in [2, 22^*)$.

Moreover, we need the following assumptions:

(g_1) Let $g \in C(\square^N \times \square, \square)$, $g(x_1, \cdots, x_N, s)$ be 1-periodic in x_i, $1 \le i \le N$, $g(x, s)$ is odd in s.

(g_2) there exists $4 < p < 22^*$ such that $|g(x, t)| \le C(1 + |t|^{p-1})$ for all $(x, t) \in \square^N \times \square$.

(g_3) $g(x, t) = \circ(|t|)$ uniformly in x as $|t| \to 0$ and $tg(x, t) \ge 0$ for all $(x, t) \in \square^N \times \square$.

(g_4) $\lim_{|t| \to +\infty} \dfrac{G(x, t)}{t^4} = +\infty$ uniformly in $x \in \square^N$, where $G(x, t) := \int_0^t g(x, s) ds$.

(g_5) $tg(x, t) - 4G(x, t) \ge stg(x, st) - 4G(x, st)$, $\forall (x, t) \in \square^N \times \square$, $\forall s \in [0, 1]$.

The equation (1.1) is the Euler-Lagrange equation of the energy functional

$$J(u) = \frac{1}{2} \int_{\square^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\square^N} (V(x)u^2 + 2\lambda u^2 |\nabla u|^2) dx - \frac{1}{22^*} \int_{\square^N} |u|^{22^*} dx - \int_{\square^N} G(x, u) dx.$$

For $\theta \in (0, 1]$, let $J_\theta(u) = \dfrac{1}{4} \theta \int_{\square^N} (|\nabla u|^4 + u^4) dx + J(u)$. Similar to Lemma 2.1 in [5] we can proof that J and $J_\theta \in C^1(E, \square)$.

Let $*$ denote the action of Z^N on $H^1(\square^N)$ given by $(k * u)(x) := u(x - k)$, $k \in Z^N$. Set $O(u) := \{k * u : k \in Z^N\}$. $O(u)$ is called the orbit of u with respect to the action of Z^N. Under the assuming conditions, if u is a solution of (1.1), then so is $k * u$ for all $k \in Z^N$. If $u_0 \in H^1(\square^N) \bigcap W^{1,4}(\square^N)$ is a critical point of a functional F and F is Z^N-invariant, i.e. $F(k * u) = F(u)$ for all $k \in Z^N$ and all $u \in H^1(\square^N) \bigcap W^{1,4}(\square^N)$, then $O(u_0)$ is called a critical orbit of F. Two solutions u_1, u_2 of (1.1) are said to be geometrically distinct if $O(u_1) \ne O(u_2)$.

A sequence $\{u_n\} \subset E$ is called a P. S. sequence of J if $\{J(u_n)\}$ is bounded and $J(u_n) \to 0$ in E^*. We say that J satisfies the P. S. condition if every P. S. sequence possesses a convergent subsequence.

The main result of this paper is the following.

Theorem 1.1 Suppose that (V) and (g_1) - (g_5) are satisfied. Then the equation (1.1) admits infinitely many pairs $\pm u$ of geometrically distinct solutions.

Remark 1.1 For the case of non-critical growth, notice that (g_5) is weaker than the following $(g_5)'$. Hence, our Theorem 1.1, at the case $\lambda = 1$, improves Theorem1.1 in[4], and our method is different from[4]. $(g_5)'$ $\dfrac{g(x, t)}{t^3}$ is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, \infty)$.

Throughout the paper, C, c, C_i and c_i express distinct constants.

The proof of the main result

To begin with, we define, for each fixed $\theta \in (0, 1]$, $M = M(\theta) := \{u \in E \setminus \{0\} : \langle J_\theta'(u), u \rangle = 0\}$. For any $u \in E \setminus \{0\}$ and $t > 0$, set $h(t) := J_\theta(tu)$.

We divide the proof of Theorem 1.1 into the following Lemmas.

Lemma 2.1 For each $u \in E \setminus \{0\}$, there exists an unique $t_u > 0$ such that $h(t_u) = \max_{s \ge 0} h(s)$,

$h'(t) > 0$ for $0 < t < t_u$ and $h'(t) < 0$ for $t_u < t$. Moreover, $tu \in M$ if and only if $t = t_u$.

Proof. For any $\varepsilon > 0$, by (g_2) and (g_3), there exists $C_\varepsilon > 0$ such that

$$|g(x,u)| \leq \varepsilon |u| + C_\varepsilon |u|^{p-1}, |G(x,u)| \leq \varepsilon |u|^2 + C_\varepsilon |u|^p, \forall (x,u) \in \square^N \times \square. \tag{2.1}$$

Hence h has a positive maximum and there exist a $t_u > 0$ such that $h'(t_u) = 0$ and $h'(t) > 0$ for $0 < t < t_u$.

We claim that $h'(t) \neq 0$ for all $t > t_u$. Indeed, if the conclusion is false, then, from the above arguments, there exists a $t_u < t_2 < +\infty$ such that $h'(t_2) = 0$ and $h(t_u) \geq h(t_2)$. But (g_5) implies that

$$h(t_2) > \frac{1}{4} t_u^2 \int_{\square^N} \left[|\nabla u|^2 + V(x)u^2 \right] dx + \frac{1}{2N} t_u^{22^*} \int_{\square^N} |u|^{22^*} dx + \int_{\square^N} \left[\frac{1}{4} g(x,t_u u)t_u u - G(x,t_u u) \right] dx = h(t_u).$$

This is a contradiction. This claim is proved.

The second conclusion is an immediate consequence of the fact that $h'(t) = t^{-1} \left\langle J_\theta'(tu), tu \right\rangle$. This completes the proof of Lemma 2.1.

Lemma 2.2 (1)There exists a $\rho > 0$ such that $c(\theta) := \inf_{u \in M} J_\theta(u) \geq \inf_{u \in S_\rho} J_\theta(u) > 0$ and there exists a constant $\delta_0 > 0$ such that $\|u\|_E \geq \delta_0$ for all $u \in M$, where $S_\rho := \{ u \in E : \|u\|_E = \rho \}$.

(2)There exists a positive number δ_1 independent of θ such that $c(\theta) \geq \delta_1$.

Proof. For any $\rho > 0$, if $u \in E$ with $\|u\|_E \leq \rho$, then $\int_{\square^N} |u|^2 |\nabla u|^2 dx \leq \rho^4$. Hence for small $\rho > 0$ and $\varepsilon < \frac{V_0}{4}$, by $(V), (g_2), (g_3)$ and the Sobolev inequality, one has

$$J_\theta(u) \geq \frac{\theta}{4} \int_{\square^N} (|\nabla u|^4 + u^4) dx + \frac{1}{2} \int_{\square^N} (1 + 2\lambda u^2)|\nabla u|^2 dx + \frac{1}{2} \int_{\square^N} V(x)u^2 dx$$

$$- \frac{1}{22^*} \int_{\square^N} |u|^{22^*} dx - \int_{\square^N} (\varepsilon u^2 + C_\varepsilon |u|^{22^*}) dx \geq \frac{\theta}{4} \|u\|_{W^{1,4}}^4 + C_2 \|u\|_{H^1}^2 \geq C(\theta) \|u\|_E^4$$

Whenever $\|u\|_E \leq \rho$. For any $u \in M$, Lemma 2.1 implies that $J_\theta(u) = \max_{t \geq 0} J_\theta(tu)$. Take a $s > 0$ with $su \in S_\rho$. Then $J_\theta(u) \geq J_\theta(su) \geq \inf_{v \in S_\rho} J_\theta(v) \geq C(\theta)\rho^4 > 0$ and hence $c(\theta) := \inf_{u \in M} J_\theta(u) \geq \inf_{v \in S_\rho} J_\theta(v) > 0$.

Moreover, for each $u \in M$, by (g_3), one has $c(\theta) \leq C_3(\|u\|_E^4 + \|u\|_E^2)$. Hence there exists a constant $\delta_0 > 0$ such that $\|u\|_E \geq \delta_0$ for all $u \in M$.

Further, for any $\rho > 0$, if $u \in E$ with $\alpha(u) := \|u\|_{H^1} + (\int_{\square^N} u^2 |\nabla u|^2 dx)^{\frac{1}{4}} \leq \rho$, then $\int_{\square^N} u^2 |\nabla u|^2 dx \leq \rho^4$. Hence for small $\rho_0 > 0$ and $\varepsilon < \frac{V_0}{4}$, by $(V), (g_2), (g_3)$ and the Sobolev inequality, one has

$$J_\theta(u) \geq C(\|u\|_{H^1}^2 + \int_{\square^N} u^2 |\nabla u|^2 dx) \text{ whenever } \alpha(u) \leq \rho_0. \text{ For any } u \in M, \text{ Lemma 2.1 implies that}$$

$J_\theta(u) = \max_{t \geq 0} J_\theta(tu)$. Take a $s > 0$ with $\alpha(su) = \rho_0$. Then $J_\theta(u) \geq J_\theta(su) \geq \frac{C}{2} \rho_0^4 := \delta_1 > 0$, hence $c(\theta) \geq \delta_1$. This completes the proof of Lemma 2.2.

Lemma 2.3 J_θ is coercive on M, i.e. $J_\theta(u) \to +\infty$ as $\|u\|_E \to \infty, u \in M$.

Proof. Arguing by contradiction, suppose there exists a sequence $\{u_n\} \subset M$, such that $\|u_n\|_E \to \infty$ and $J_\theta(u_n) \leq d$ for some $d < 0$. By (g_5) and (V), one has

$$d \geq J_\theta(u_n) - \frac{1}{4} \left\langle J_\theta'(u_n), u_n \right\rangle \geq \frac{1}{4} \min\{1, V_0\} \int_{\square^N} (|\nabla u_n|^2 + |u_n|^2) dx + \frac{1}{2N} \int_{\square^N} |u_n|^{22^*} dx.$$

Hence $\{\|u_n\|_{H^1}\}$ and $\{\|u_n\|_{L^{22^*}(R^N)}\}$ are bounded. By interpolation, $\{u_n\}$ is bounded in $L^s(\square^N), 2 \leq s \leq 22^*$.

Consequently, by (2.1), there exists a constant $C_0 > 0$ such that $\left|\int_{\square^N} G(x, u_n) dx\right| \leq C_0$, hence

$J_\theta(u_n) \geq \dfrac{\theta}{4}\|u_n\|_{W^{1,4}}^4 - C$, which implies $\dfrac{\theta}{4}\|u_n\|_{W^{1,4}}^4 \leq d + C < \infty$. Consequently $\{\|u_n\|_E\}$ is bounded, a contradiction. This ompletes the proof of lemma2.3.

Lemma 2.4 Let $D \subset E \setminus \{0\}$ be a compact subset. Then there exists a $R > 0$ such that $J_\theta < 0$ on $(\square^+ D) \setminus B_R(0)$, where $B_R(0) := \{u \in E : \|u_n\|_E < R\}$.

Proof. Arguing by a contradiction, suppose that there exist equences $\{u_n\} \subset D$ and $\{t_n\} \subset \square^+$ such that $J_\theta(t_n u_n) \geq 0$ and $t_n \to +\infty$ as $n \to \infty$. By the compactness of D, we may assume that $u_n \to u \in D$ and $\|u_n\|_E \leq C$. Hence $0 \leq \dfrac{J_\theta(t_n u_n)}{t_n^4} \leq C_1 - \dfrac{1}{22^*} t_n^{2(2^*-2)} \int_{\square^N} |u_n|^{22^*} dx \to -\infty$, a contradiction. This completes the proof of Lemma 2.4.

Let S be the unit sphere in E. Define a mapping $m = m(\theta) : S \to M$ and a functional $\Psi = \Psi(\theta) : S \to \square$ by $m(u) = t_u u$ and $\Psi(u) = J_\theta(m(u))$, where t_u is as in Lemma 2.1. By Lemma 2.1, 2.2 and 2.4, similar to Lemmas 3.6-3.8 in [4], we can prove the following Lemmas 2.5-2.7.

Lemma 2.5 The mapping m is a homeomorphism between S and M, and the inverse of m is given by $m^{-1}(u) = \dfrac{u}{\|u\|_E}$.

Lemma 2.6 (1) $\Psi \in C^1(S, \square)$ and for each $u \in S$, one has $\langle \Psi'(u), z \rangle = \|m(u)\|_E \langle J'_\theta(m(u)), z \rangle$ for all $z \in T_u(S)$, where $T_u(S)$ is the tangent space of S at point u.

(2) If $\{u_n\}$ is a Palais-Smale sequence of Ψ, then $\{m(u_n)\}$ is a Palais-Smale sequence of J_θ. If $\{u_n\} \subset M$ is a bounded Palais-Smale sequence of J_θ, then $\{m^{-1}(u_n)\}$ is a Palais-Smale sequence of Ψ.

(3) u is a critical point of Ψ if and only if $m(u)$ is a nontrivial critical point of J_θ. Moreover, the corresponding values of Ψ and J_θ coincide and $\inf_S \Psi = \inf_M J_\theta$. (4) If J_θ is even, then so is Ψ.

Lemma 2.7 The mapping m^{-1} defined in Lemma 2.5 is Lipschitz continuous.

Lemma 2.8 $c_k(\theta)$ is a critical value of Ψ.

Proof. If $c_k(\theta)$ is not a critical value of Ψ, then for any $w \in S$, one has $\Psi(w) \neq c_k(\theta)$ or $\Psi'(w) \neq 0$. Hence there exists $\delta > 0$ such that $N_{C_k(\theta),\delta} := \{w \in S : |\Psi(w) - c_k(\theta)| < \delta, \|\Psi'(w)\|_{E^*} < \delta\} = \varnothing$. Otherwise, there exists a sequence $\{w_n\} \subset S$ such that $\Psi(w_n) \to c_k(\theta)$ and $\|\Psi'(w_n)\|_{E^*} \to 0$. Set $u_n = m(w_n)$. Then, $\{u_n\} \subset M$ is a $(PS)_{c_k(\theta)}$ sequence of J_θ. By Lemma 2.3 the sequence $\{u_n\}$ is bounded in E. Hence, up to a subsequence, one has $u_n \xrightarrow{\text{weak}} u$ in $W^{1,4}(\square^N)$ and $H(\square^N)$. Notice that $22^* < \dfrac{4N}{N-2}$, one has $u_n \to u$ in $L_{loc}^s(\square^N)$ for all $s \in [2, 22^*)$ and $u_n(x) \to u(x)$ a.e. $x \in \square^N$. By weakly convergence, one has $\int_{\square^N} |\nabla u_n|^2 dx \geq 2\int_{\square^N} \nabla u_n \cdot \nabla u dx - \int_{\square^N} |\nabla u|^2 dx = \int_{\square^N} |\nabla u|^2 dx + o_n(1)$

and $\quad o_n(1) = \langle J'_\theta(u_n) - J'_\theta(u), u_n - u \rangle$

$$\geq C\theta\|u_n - u\|_{W^{1,4}}^4 + C\|u_n - u\|_{H^1}^2 + 2\lambda \int_{\square^N} (|u_n|^2 - |u|^2) \cdot \nabla u \cdot \nabla(u_n - u) dx$$

$$+ 2\lambda \int_{\square^N} (|\nabla u_n|^2 - |\nabla u|^2) \cdot u \cdot (u_n - u) dx - \int_{\square^N} \left[|u_n|^{22^*-2} u_n - |u|^{22^*-2} u\right] \cdot (u_n - u) dx$$

$$- \int_{\square^N} [g(x, u_n) - g(x, u)] \cdot (u_n - u) dx$$

where $o_n(1) \to 0$ as $n \to \infty$. Hence, by Hölder inequality and the boundedness of $\{u_n\}$ in E, one

has
$$C\theta\|u_n-u\|_{W^{1,4}}^4+C\|u_n-u\|_{H^1}^2\le o_n(1)+C_1\|u_n-u\|_{22^*}^2+\int_{\square^N}\left(|g(x,u_n)|+|g(x,u)|\right)\cdot|u_n-u|\,dx.\qquad(2.2)$$

For any $\varepsilon>0$, by $(g_2),(g_3)$ and the boundedness of $\{u_n\}$ in E, there exist constants $C>0$ and $C_\varepsilon>0$ such that

$$\int_{\square^N}\left(|g(x,u_n)|+|g(x,u)|\right)|u_n-u|\,dx\le\varepsilon C\|u_n-u\|_E+C_\varepsilon\|u_n-u\|_{22^*}.\qquad(2.3)$$

Combining (2.2) and (2.3) we know that for any $\varepsilon>0$, there exist positive constants C and C_ε such that

$$C\theta\|u_n-u\|_{W^{1,4}}^4+C\|u_n-u\|_{H^1}^2\le o_n(1)+\varepsilon C\|u_n-u\|_E+C_\varepsilon\|u_n-u\|_{22^*}.\qquad(2.4)$$

If $\|u_n-u\|_{22^*}$ does not go to 0 as $n\to\infty$, then, by Lemma I.1 in [7], up to a subsequence, there exists $\varepsilon_0>0$ such that $\sup\limits_{y\in\square^N}\int_{B_1(y)}|u_n-u|^2\,dx\ge\varepsilon_0$. Hence there exists $y_n\in\square^N$ such that

$\int_{B_1(y_n)}|u_n-u|^2\,dx=\max\limits_{y\in\square^N}\int_{B_1(y)}|u_n-u|^2\,dx\ge\varepsilon_0$. By the assumptions of periodicity, we can assume $\{y_n\}$ is bounded in $\square^N$. Consequently, there exists a bounded domain $\Omega\subset\square^N$ such that $\int_\Omega|u_n-u|^2\,dx\ge\int_{B_1(y_n)}|u_n-u|^2\,dx\ge\varepsilon_0$, a contradiction. Hence $\|u_n-u\|_{22^*}\to0$ as $n\to\infty$. Consequently, by (2.7), $u_n\to u$ in E, and hence u is a critical point of J_θ and $J_\theta(u)=c_k(\theta)$. Moreover, by Lemma 2.7 we know that $u\in M$, and hence Lemma 2.5 implies $w:=m^{-1}(u)\in K_{c_k(\theta)}$, a contradiction. This shows $N_{c_k(\theta)}=\varnothing$. Therefore, by Remark II.3.12 in [13], there exists $\varepsilon_0>0$ such that for any $0<\varepsilon<\bar\varepsilon\le\varepsilon_0$, there exists a continuous 1-parameter family of homeomorphisms $\eta(t,\cdot)$ of S, $0\le t<\infty$, with the properties: (1^0) $\eta(w,t)=w$, if $t=0$, or $\Psi'(w)=0$, or $|\Psi(w)-c_k(\theta)|\ge\bar\varepsilon$; (2^0) $\Psi(\eta(w,t))$ is non-increasing in t for any $w\in S$; (3^0) $\eta(\Psi^{c_k(\theta)+\varepsilon},1)\subset\Psi^{c_k(\theta)-\varepsilon}$; (4^0) $\eta(\cdot,s)\circ\eta(\cdot,t)=\eta(\cdot,s+t)$ for all $s,t\ge0$; (5^0) $\eta(w,t)$ is odd in w for $t\ge0$.

Moreover, by $N_{c_k(\theta),\delta}=\varnothing$ we know that there exists $0<\varepsilon_1<\varepsilon_0$ such that $\Psi^{c_k(\theta)+\varepsilon_1}_{c_k(\theta)-\varepsilon_1}\bigcap K=\varnothing$. For each $w\in\Psi^{c_k(\theta)+\varepsilon_1}$, by the property (3^0) of η we know that $\Psi(\eta(w,1))\le c_k(\theta)-\varepsilon_1$. Let $e=e(w)$ be the infimum of the time for which $\Psi(\eta(w,t))\le c_k(\theta)-\varepsilon_1$. It is easy to see that $e:\Psi^{c_k(\theta)+\varepsilon_1}\to[0,+\infty)$ is a continuous mapping. Since Ψ is even, so is e. Define a mapping $h:\Psi^{c_k(\theta)+\varepsilon_1}\to\Psi^{c_k(\theta)-\varepsilon_1}$ by $h(w):=\eta(w,e(w))$. Then h is odd and continuous. It follows from the mapping property of the genus and the definition of $c_k(\theta)$ that $k\le\gamma(\Psi^{c_k(\theta)+\varepsilon_1})\le\gamma(\Psi^{c_k(\theta)-\varepsilon_1})\le k-1$ a contradiction. This completes the proof of Lemma 2.8.

Lemma 2.9 Let $\{\theta_n\}\subset(0,1]$ be such that $\theta_n\to0$. Let $u_n\in E$ be a critical point of J_{θ_n} with $J_{\theta_n}(u_n)\le c$ for some constant c independent of n. Then, up to a subsequence, we have $u_n\xrightarrow{weak}u$ in $H^1(\square^N)$, $u_n\nabla u_n\xrightarrow{weak}u\nabla u$ in $L^2(\square^N)$, $u\in H^1(\square^N)\bigcap L^\infty(\square^N)$ is a critical point of J and $J(u)\le\liminf\limits_{n\to\infty}J_{\theta_n}(u_n)$.

Proof. As the proof of Lemma 2.3 we can prove that the sequences $\{\|u_n\|_{H^1}\}$ $\{\|u_n\|_{22^*}\}$ $\{\int_{\square^N}|u_n|^2|\nabla u_n|^2\,dx\}$ $\{\int_{\square^N}G(x,u_n)\,dx\}$ and $\{\theta_n\|u_n\|_{W^{1,4}}^4\}$ are bounded. Hence, by the Sobolev embedding and the interpolation, up to a subsequence, one has $u_n\xrightarrow{weak}u$ in $H^1(\square^N)$, $u_n\to u$ in $L^s_{loc}(\square^N)$ for $s\in[2,22^*)$, $u_n(x)\to u(x)$ a.e. $x\in\square^N$. Up to a subsequence, $\lim\limits_{n\to\infty}\int_{\square^N}|u_n-u|^s\,dx=0$ $\forall s\in[2,22^*)$.

By using Moser iteration we can prove $u\in L^\infty(\square^N)$. Hence, by approximation we obtain $\langle J'(u),\varphi\rangle=0$, $\forall\varphi\in H^1(\square^N)\bigcap L^\infty(\square^N)$. This shows that u is a critical point of J.

Moreover, one has
$$\liminf_{n \to \infty} J_{\theta_n}(u_n) \ge \frac{a}{4}\int_{\square^N}|\nabla u|^2\,dx + \frac{1}{4}\int_{\square^N}V(x)|u|^2\,dx + (\frac{1}{4}-\frac{1}{22^*})\int_{\square^N}|u|^{22^*}\,dx$$
$$+\int_{\square^N}[\frac{1}{4}ug(x,u)-G(x,u)]dx = J(u) - \frac{1}{4}\langle J'(u),u\rangle = J(u).$$

This completes the proof.

Proof of Theorem 1.1 Take a sequence $\{\theta_n\} \subset (0,1)$ with $\theta_n \downarrow 0$. Then for any fixed $k \in \square$, by Lemma 2.9 we know that there exists a sequence $\{v_k(n) : n = 1,2,\cdots\}$ of critical points for Ψ such that $\Psi(v_k(n)) = c_k(\theta_n)$. Hence, for each $n \in \square$, Lemma 2.6 implies that $\{u_k(n) := m(v_k(n))\}$ is a nontrivial critical point of J_{θ_n} and $J_{\theta_n}(u_k(n)) = c_k(\theta_n)$. Notice that $\delta_1 \le c_k(\theta_n) \le c_k(1)$. Up to a subsequence, we have, as $n \to \infty$, $v_k(n) \to v_k$ $u_k(n) \to u_k = m(v_k)$ in $H^1(\square^N)$, $u_k(n)\nabla u_k(n) \to u_k\nabla u_k$ in $L^2(\square^N)$, $\theta_n\int_{\square^N}(|\nabla u_k(n)|^4 + |u_k(n)|^4)dx \to 0$, $J_{\theta_n}(u_k(n)) \to J(u_k) := c_k$ and $u_k \in H^1(\square^N) \bigcap L^\infty(\square^N)$ is a nontrivial critical point of J. By Lemma 2.6(3), v_k is a critical point of Ψ.

Now, set $\tilde{K} := \bigcup_{k=1}^{\infty}K_{c_k}$. Choose a subset F of $\tilde{K}$ such that $F = -F$ and each orbit $O(u) \subset \tilde{K}$ has a unique representative in F. As proof of Lemma 2.11 in [15] we can prove that F is an infinite set. Combining this with Lammas 2.6(3), we deduce Theorem 1.1.

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