# Image Denoising Based on TVD Runge-Kutta Method

# And Projection Algorithm

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## Abstract

In this paper, based on Chambolle's Projection Algorithm, an PDE was proposed to replace the total variation model for removing image noise. In order to improve the algorithm, the TVD Runge-Kutta method, which has high-precision of time direction, was introduced to solve the equation. The numerical results show that the algorithm has high efficiency. It can make the image smoother, and it improve the speed of image denoising.
*Keywords: Total variation，Image denoising，Projection gradien*

## 1.   Introduction

The image has become an important source for people to get the objective world, but the actual image often disturbed by the noise in the transmission, thus it brought to difficulties in image analysis and processing, and affecting the image segmentation, edge detection and feature extraction and so on. Denoising is the problem that only takes into account the random phenomenon and it consists of removing noise from an image. The most commonly studied noise model is additive white Gaussian noise, where the observed noisy image $u_0$ is related to the underlying true image $u$ by the degradation model [1]

$$u_0 = u + \eta \tag{1}$$

$u$ is an original image, $u_0$ observed image, $\eta$ is the standard deviation of the noise. We recover the original image $u$ from $u_0$ is a typicall ill posed inverse problem. The classic solution method is the use of regularization methods, set up the corresponding variational model [2 ]:

$$F(u) = \int_\Omega |u_0 - Ru|^2 dx + \lambda \int_\Omega |\nabla u|^2 dx \tag{2}$$

The first term $\int_{\Omega}|u_0 - Ru|^2 dx$ is measures the fidelity to the data, and the

second $\lambda\int_{\Omega}|\nabla u|^2 dx$ is a smoothing term. The $\lambda$ is the regularization

parameter. In order to preserve the edges as much as possible, One of the most models for image reconstruction is the total variation, which is based model developed by Rudin, Osher and Fatemi [3]. This model studies the minimizer of the following energy:

$$F(u) = \int_{\Omega}|u_0 - Ru|^2 dx + \lambda\int_{\Omega}\varphi(|\nabla u|)dx \qquad (3)$$

The next problem is get a numerical approximation for total variation denoising, if we directly the Euler-Lagrange equations, the difficulty is to define variations on $BV(\Omega)$ have been developed. In order to circumvent this

difficulty, Chambolle has proposed projection algorithm[4]. When $\phi(t) = t$, and

$R$ is the identity operator, Chambolle and Lions proved that (2)-(3) is naturally linked to the following unconstrained minimization problem:

$$\min\frac{1}{2\lambda}\int_{\Omega}|u_0 - u|^2 dx + \int_{\Omega}|\nabla u|dx \qquad (4)$$

In view of (4), denoising is performed as an infinite-dimensional minimization problem, where the search space is the set of all images with bounded variation. BV-functions appear as a natural model for images, characterized by the appearance of discontinuous hypersurface. Let $\Omega \in R^N$ be a bounded domain, $N \geq 2$, we can observe that the total variation of a function $u \in L^1(\Omega)$ is defined by

$$TV(u) = \sup\left\{\int_{\Omega}u(x)div\varphi(x)dx : \varphi = (\varphi_1, \varphi_2, ..., \varphi_N) \in C_c^1(\Omega; R^2), |\varphi(x)| \leq 1, \forall x \in \Omega\right\}$$

Where $div\varphi(x) = \sum_{i=1}^{N}\frac{\partial\varphi_i}{\partial x_i}$, It is well known [5] that $TV(u)$ is finite if and

only if its distributional derivative D$u$ is a finite Radon measure, and we have $TV(u) = |D(u)|$.

Many algorithms for total variation denoising have been developed, in this paper we focus on Chambolle's projection algorithm for minimizing the total variation of a gray-scale image. It is based on a dual formulation and it is related to the works of Chan, Golub and Mulet [6], Prime-Dual method. We introduce the discrete gradient operator and discrete divergence operator with the central difference. On computing side, we proposed TVD Runge-Kutta method to numerical solution process.

## 2. Projection algorithm based on the Central Difference

In order to improve the image denoising effect , minimize the total variable projection can be seen as a problem in an appropriate Convex set. On the computing side, the most commonly used discrete variational model is based on the discrete energy .

Let us fix our notations. To simplify, our images will be a two-dimensional matrix of size $N \times N$ , $u_{0i,j}$ $(i, j = 1, 2, 3...N)$ is discrete imags , $X = R^{N \times N}$ is a set of the all discrete images. For $u_{0i,j} \in X$ , we introduce the discrete gradient operator $\nabla U : X \rightarrow X \times X$ ,via the central difference ,

$$\nabla u_1 = \begin{cases} \dfrac{u_{i+1, \ j} - u_{i-1,j}}{2}, & 1 < i < N \\ 0 , & i = 1, N \end{cases} , \nabla u_2 = \begin{cases} \dfrac{u_{i,j+1} - u_{i,j-1}}{2}, & 1 < j < N \\ 0 , & j = 1, N \end{cases}$$

That is, for every $p \in Y$ and $u \in X$ , $\langle -divp, u \rangle_X = \langle p, \nabla u \rangle_Y$ .

One checks easily that div is given by

$$(divp)_{i,j} = (divp)^x_{i,j} + (divp)^y_{i,j}$$

$$divp_1 = \begin{cases} \dfrac{1}{2} p(i+1, j) & i = 1, 2 \\ \dfrac{p_{i+1, \ j} - p_{i-1,j}}{2}, & 2 < i < N - 1 \\ -\dfrac{1}{2} p(i-1, j) & i = N - 1, N \end{cases} , divp_2 = \begin{cases} \dfrac{1}{2} p(i, j+1) & j = 1, 2 \\ \dfrac{p_{i, \ j+1} - p_{i,j-1}}{2}, & 2 < j < N - 1 \\ -\dfrac{1}{2} p(i, j-1) & j = N - 1, N \end{cases}$$

thus, the discrete variational model is:

$$J(u) = \sum_{i,j=1}^{N} \left| (\nabla u)_{i,j} \right|$$

where， function $u \in L_1(\Omega)$ . the problem is converted to：

$$\min_{u \in X} \{ J(u) + \frac{1}{2\lambda} |u - u_0|_{X^2} \} \tag{6}$$

where $u$ , $u_0$ are discretization vectors of related continuous variables, $\|.\|_X^2$ is the Euclidean norm in X, is given by $\|u\|_X^2 = \langle u, u \rangle_X$ .The minimization of the model (5) can be performed using arguments from convex analysis. Under this scope, we present the projection approach of Chambolle in the continuous setting. It is shown in [4] that the solution $u$ of problem (5) is given by

$$u = f - \pi_{\lambda K_F(f)}$$

Computing the nonlinear projection $\pi_{\lambda K_F(f)}$ equivalent to sovling the following problem:

$$\min\{\|\lambda divp - u_0\|^2 : p \in Y, |p_{i,j}|^2 - 1 \le 0, \forall i,j = 1,...,N\} \quad (6)$$

Then obtained from the KKT conditions[7],there existence a Lagrange multiplier $\alpha_{i,j} \ge 0$ ,associated to each constraint in problem (6), such that we hanve for each $i, j$

$$-(\nabla(\lambda divp - u_0))_{i,j} + \alpha_{i,j} p_{i,j} = 0$$

Where either $\alpha_{i,j} \ge 0$ and $|p_{i,j}| = 1$ ,or $|p_{i,j}| < 1$ ,and $\alpha_{i,j} = 0$ .Thus in any case $\alpha_{i,j} = |(\lambda divp - u_0)_{i,j}|$ .

Then chambolle propses the fixed point algorithm[8],choose $\tau > 0$ ,let $p^0 = 0$ and for any $n \ge 0$ ,

$$p_{i,j}^{n+1} = p_{i,j}^n + \tau((\nabla(\lambda divp^n - u_0/\lambda))_{i,j} - |(\nabla(\lambda divp^n - u_0/\lambda))_{i,j}| p_{i,j}^{n+1}),$$

So that

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau(\nabla(\lambda divp^n - u_0/\lambda))_{i,j}}{1 + \tau|(\nabla(\lambda divp^n - u_0/\lambda))_{i,j}|} \quad (7)$$

After a number of iteration steps to meet the conditions image $p$ ,then we can obtain the denoising image $u$ from $u = u_0 - \lambda divp$ .

**Remark :** When $\tau \le \frac{1}{2}, \lambda divp^n$ converges to $\pi_{\lambda k}(g)$ as $n \to \infty$ .

It is shown in [4] that by definition of the operator div with $K$ given by

$$\{div\ p : p \in Y, |p_{i,j}| \le 1, \forall i,j = 1,...,N\}$$

Now, we let

$$p_{0,j} = p_{1,j} = p_{N,j} = p_{N-1,j} = 0, p_{i,0} = p_{i,1} = p_{i,N} = p_{i,N-1} = 0,$$

Hence $K^2 \le 2$ .

## 3. Tuning $\lambda$

The choice of the parameter $\lambda$ affects the balance between removing the noise and preserving the edges. Thus we must choose a optimum parameters . For TV denoising, there are some algorithms to tune a value of $\lambda$ , Chambolle [ ] propses

that we can replace $\lambda$ with the new value $N\sigma / \|u - u_0\|_X$ .We initialize the iteration process with the following empirical estimate of $\lambda$ , suggested in [9]:

$$\lambda_0 = \frac{0.7079}{\sigma} + \frac{0.6849}{\sigma^2},$$

And the algorithm is described as follows:

Input $\sigma$ and $\lambda_0$ ,
For i=1:30;
$\lambda = \lambda_0$ ;
$u = u_0 - \lambda divp$ ;
$\lambda = N\sigma / \|u - u_0\|_X$ ;
end

## 4. The projection based on TVD Runger-Kutta algorithm

Discrete time direction generally using the standard Euler method , but only linear accuracy Euler method to obtain high accuracy, usualy we use TVD Runge-Kutta methods [10]，TVD Runge-Kutta method is the traditional Runge-Kutta method to transform Runge-Kutta，which its weights are positive. In order to avoid unnecessary oscillation generated numerical solution process. We use the third-order TVD Runge-Kutta format:

$$u^{(1)} = u^n + \Delta t L(u^n)$$

$$u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)})$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)})$$

For the discretization equation(7) can be turned into the equation about $\frac{\partial p}{\partial t}$ ,

that is as the following form：

$$L(p) = \frac{\partial p}{\partial t} = \left( \nabla \left( \lambda divp^n - \frac{u_0}{\lambda} \right) \right) - \left| \nabla \left( \lambda divp^n - \frac{u_0}{\lambda} \right) \right| p$$

(8)

By the third-order TVD Runge-Kutta, we can obtain the improved algorithm iteration steps are as follows :

$$p_1 = p^n + \Delta t \times L(p^n)$$

$$p_2 = \frac{3}{4} p^n + \frac{1}{4} p_1 + \Delta t \times L(p_1)$$

$$p^{n+1} = \frac{1}{3} p^n + \frac{2}{3} p_2 + \frac{2}{3} \times \Delta t \times L(p_2)$$

$$u = u_0 - \lambda divp$$

## 5. Experimental Results

In this section, we take several groups experimental to prove the effectiveness of our model. We use root of mean square of errors(RMSE）and peak signal to noise ratio(PSNR) to compare algorithm the effect of denoising. We define the PSNR and RMSE as:

$$PSNR = 10\lg \frac{\max |u_{ij}|^2}{M \times N \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} (u_{ij} - u_{0ij})^2}$$

$$RMSE = \sqrt{\frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left( u(m,n) - u_0(m,n) \right)^2}$$

We choose two different standard images to test. We try to choose the regularization parameter by QPSO method. We can demonstrate how to choose the value of paramter by our method, which is influences the denoising result.

In the experiment, We added the Gaussian noise of satandard deviation is 15, 25, 35, 50 for each image. We set the the iterations is 50.

In order to display our algorithm's outperformance, we first use Chambolle's projection algorithm to solve the ROF model.Then we initialized the iteration with the constant paramter $\lambda = 0.1$. Now we can see how the QPSO algorithm for TV-regularized denoising behaves.

When we add different satandard deviation $\sigma$ in lena image, the optimal regularization parameter $\lambda$ is different in our method. It has changed due to QPSO with the increasing of iteration. As the deviation is increasing, the paramter is decreasing.

As shown in Fig.1-4, we first set the regularization parameter $\lambda = 0.1$ in ROF model with a constant value. When we add different satandard deviation $\sigma$ in lena image, the tracking of the regularization parameter $\lambda$ is different in our method. Especially, from the vision the denoising image effect is much better than the using the constant parameter. We observe that while sharp edges or some detail have been blurred or disappeared in the method by constant paramter. But in our method we have been able to remove much of the noise while preserving the while sharp edges. The first example demonstrates the total

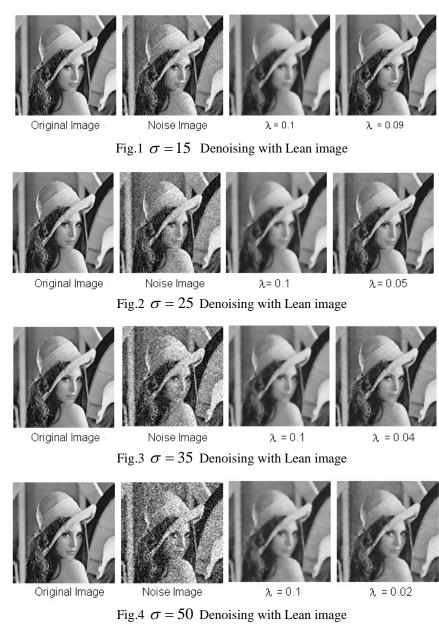variation image denoising based QPSO for Gaussian denoising affect the value of $\lambda$ influences the result.



Fig.1 $\sigma = 15$ Denoising with Lean image



Fig.2 $\sigma = 25$ Denoising with Lean image



Fig.3 $\sigma = 35$ Denoising with Lean image



Fig.4 $\sigma = 50$ Denoising with Lean image

Table 1 The values of RMSE for Lena image

| | σ=50 | | σ=35 | | σ=25 | | σ=15 | |
|---|---|---|---|---|---|---|---|---|
| | $\lambda$ | PSNR | $\lambda$ | PSNR | $\lambda$ | PSNR | $\lambda$ | PSNR |
| Constant paramter | 0.1 | 30.50 | 0.1 | 31.03 | 0.1 | 31.29 | 0.1 | 31.50 |
| Optimal parameter | 0.02 | 30.82 | 0.04 | 31.91 | 0.05 | 33.01 | 0.09 | 34.80 |

In the Table1, every denoising image's PSNR is shows in the table, we clearly see the PSNR is better than using constant parameter.

## 6.  Conclusions

In this paper, we maily using intelligent optimization algorithm to the inverse problem in image processing. The results of the regularization parameter are better than using the constant value which is formerly chosen. It is because of the fact that whenever QPSO wants to estimate the regularization parameter, it tries to eliminate the smoothing effect by choosing a small parameter and then tries to eliminate the noise effect by choosing a big regularization parameter. In the future, different types of inverse problem in image processing will be tested with optimal regularization method.

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