Countable compactness in generalized L-topological spaces

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Abstract.

In this paper, the generalized countable L-compact sets and generalized Lindelöf sets are introduced in generalized L-topological spaces, based on the notion of generalized L-compactness. They are described by cover form and finite intersection property. They are preserved under generalized L-continuous mapping, inherited for L-closed subsets, and finitely additive. And an L-subset is generalized L-compact if and only if it is generalized Lindelöf and generalized countably L-compact.

Keywords: generalized L-topology, generalized countable L-compactness, generalized Lindelöf set

1. Introduction and Preliminaries

In[2], Bai introduced the concept of generalized L-topological spaces, and studied the basic concepts and basic properties in generalized L-topological spaces. Following the lines of [2], in [3], Bai introduced generalized L-compactness. In this paper, our aim is to continue the research of generalized countable L-compact sets and generalized Lindelöf sets in generalized L-topological spaces.

Throughout this paper, $(L, \vee, \wedge, ')$ is a completely distributive De Morgan algebra and X is a nonempty set. L^X is the set of all L-fuzzy sets on X. The smallest element and the largest element of L^X will be denoted by 0 and 1 respectively. The set of non-unit prime elements[4] in L is denoted by $pr(L)$. The set of nonzero co-prime elements[4] in L and L^X is denoted by

$M(L)$ and $M^*(L)$ respectively. Clearly, $r \in pr(L)$ iff $r' \in M(L)$ .The greatest minimal family of a in L is denoted by $\beta(a)$ .The greatest maximal family of a in L is denoted by $\alpha(a)$ [5,8]. Moreover for a in L, define $\beta^*(a) = \beta(a) \cap M(L)$ and $\alpha^*(a) = \alpha(a) \cap pr(L)$ .For each $\psi \subset L$,we define $\psi' = \{A' : A \in \psi\}$. For $r \in L$, $\varepsilon_r(A) = \{x \in X : A(x) \geq r\}$.

Definition1.1.[2]. Let L be a completely distributive De Morgan algebra, X be a nonempty set and δ be a collection of subsets of L^X .Then δ is called a generalized L-topology (briefly GL-t) on X if $0 \in \delta$ and $G_i \in \delta$ for $i \in I \neq \varnothing$ implies $G = \vee_{i \in I} G_i \in \delta$ .We call the pair(L^X, δ) a generalized L-topological space (briefly GL-ts) on X .The element of δ are called generalized L-open sets (briefly GL-open sets)and the complements are called generalized L-closed sets (briefly GL-closed sets).We say δ is strong if $1 \in \delta$.

Definition1.2.[6]. Each mapping $f : X \to Y$ induces a mapping $f_L^\to : X \to Y$ (called an L-valued Zadeh function or an L-fuzzy mapping or an L-forward power set operator),which is defined by
$$f_L^\to(A) = \vee\{A(x) \big| f(x) = y\} \quad (\forall A \in L^X, y \in Y)$$
.The right adjoint to $f_L^\to$ (called L-backward power set operator)is denoted $f_L^\leftarrow$ and given by $f_L^\leftarrow(B) = \vee\{A \in L^X \big| f_L^\to(A) \leq B\} = B \circ f \ (\forall B \in L^X)$.

Definition1.3.[2]. Let(L^X, δ)and(L^Y, τ)be two GL-ts's and $f^\to : L^X \to L^Y$ an L-fuzzy mapping. $f^\to$ is called a generalized L-continuous mapping(briefly GL-continuous mapping)if $f^\leftarrow(B) \in \delta$ for each $B \in \tau$.

Definition1.4[2]. Let(L^X, δ) be a GL-ts and $x_\lambda \in M^*(L^X)$. $A \in \delta'$ is

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called a generalized L- closed remote-neighborhood(briefly GLC-RN)of x_λ ,if $x_\lambda \le A$. $B \in L^X$ is called a generalized L-remote-neighborhood(briefly GL-RN)of x_λ if there is a GLC-RN A of x_λ such that $B \le A$ .The set of all GLC-RNs(GL-RNs)of x_λ is denoted by $\eta^-(x_\lambda)$ ($\eta(x_\lambda)$).

Definition1.5[2]. Let(L^X,δ)be a GL-ts, $A \in L^X$ and $\alpha \in M(L)$. $\phi \subset \delta'$ is called an α -closed -remote neighborhood family of A (briefly α -C-RF of A)if for each x_α in A ,there exists a $P \in \phi$ such that $P \in \eta(x_\alpha)$. ϕ is called an α^- -C-RF of A .

Definition1.6[2]. Let(L^X,δ)be a GL-ts and $A \in L^X$. A is called generalized L-compact(briefly GL-compact)if every α -C-RF ϕ of A has a finite subfamily which is an α^- -C-RF of A ($\alpha \in M(L)$).
(L^X,δ)is called GL-compact if 1_X is GL-compact.

2.Generalized countable L-compactness

Definition2.1. Let (L^X,δ) be a GL-ts and $A \in L^X$. A is called generalized countably L-compact if every countable α -C-RF ϕ of A has a finite subfamily which is an α^- -C-RF of A ($\alpha \in M(L)$).(L^X , δ)is called generalized countably L-compact if 1_X is generalized countably L-compact.

From the Definitions 2.1 and 1.6 we immediately obtain the following results.

Corollary 2.2. Every generalized L-compact set is generalized countably L-compact.

Definition2.3. Let (L^X,δ) be a GL-ts, $A \in L^X$ and $r \in pr(L)$. $\mu \subset \delta$ is called an r -cover of A if

for each $x \in \varepsilon_{r'}(A)$,there exists an $U \in \mu$ such that $U(x) \le r$. μ is called an r^+ -cover of A if there exists a $t \in \alpha^*(r)$ such that μ is a t -cover of A .

Theorem2.4. Let (L^X,δ) be a GL-ts and $r \in pr(L)$. $A \in L^X$ is generalized countably L-compact if and only if every countable r -cover μ of A has a finite subfamily ν which is an r^+ -cover of A .

Proof. Let A be generalized countably L-compact, μ a countable r-cover of A and $r \in pr(L)$. Put

$\phi = \mu'$, then $\phi \subset \delta'$ and for each $x \in \varepsilon_{r'}(A)$ there exists a $Q = U' \in \phi$ such that $U(x) \le r$, i.e. $r' \le Q(x)$.

Since $r \in pr(L)$, $r' \in M(L)$. By $x_{r'} \le Q$ we have $Q \in \eta(x_{r'})$, hence ϕ is a countable r'-C-RF of A. Since A is generalized countably L-compact, there is a finite subfamily v of μ such that $\psi = v'$

is an $(r')^-$-C-RF of A, i.e. for some $t \in \beta^*(r')$ and each $x \in \varepsilon_{r'}(A)$, there is a $V'(x) \in v$ such that $t \le V'(x)$, equivalently, for some $t' \in \alpha^*(r)$ and each $x \notin \varepsilon_{r'}(A)$, there is a $V(x) \in v$ such that $V(x) \le t'$. Thus μ has a finite subfamily v which is an r^+-cover of A.

Conversely, suppose every countable r-cover μ of A has a finite subfamily is an r^+-cover of A. Let ϕ be a countable α-C-RF of A, $\mu = \phi'$ and $r = \alpha'$. Since $\alpha \in M(L)$, $r \in pr(L)$. With the method of dual above, it is easily to prove that μ is a countable r-cover of A. Suppose v is a finite subfamily of

μ such that v is an r^+-cover of A. Put $\psi = v'$, then ψ is an α^--C-RF of A. Thus A is generalized countably L-compact.

Definition 2.5. Let (L^X, δ) be a GL-ts, $A \in L^X$, $r \in pr(L)$ and $\mu \subset L^X$. If for every finite subfamily

v of μ and for each $t \in \alpha^*(r)$, there is an $x \in \varepsilon_{t'}(A)$ such that $(\wedge v)(x) \ge t'$, then we say that μ has an

r^+-finite intersection property in A.

Theorem 2.6. Let (L^X, δ) be a GL-ts and $r \in pr(L)$. $A \in L^X$ is generalized countably L-compact if and only if every countable subfamily of GL-closed sets μ has an r^+-finite intersection property in A, and there is an $x \in \varepsilon_{r'}(A)$ such that $(\wedge \mu)(x) \ge r'$.

Proof. Let A be generalized countably L-compact. Suppose there is a prime element $e \in pr(L)$ and

Some countable subfamily of GL-closed sets μ has an e^+-finite intersection property in A, for each x

$\in \varepsilon_{r'}(A)$ such that $(\wedge \mu)(x) \ge e'$. Then there exists a $B \in \mu$ such that $B(x) \ge e'$, i.e. $B'(x) \le e$. This

shows μ' is a countable e -cover of A .By the Theorem2.4,there is a finite subfamily $\nu = \{B_1, \cdots, B_n\}$

of μ such that ν' is an e^+ -cover of A .Hence for some $t \in \alpha^*(e)$ and each $x \in \varepsilon_{r'}(A)$,there is an $B_i \in \nu$

such $\quad$ / $\quad$ that $\quad$ $B_i'(x) \le t$ $\quad$ .And/

so $\quad (\vee_{i=1}^n B_i')(x) \le t$,i.e. $(\wedge\nu)(x) = (\wedge_{i=1}^n B_i)(x) \ge t'$,which contradicts that μ has an e^+ -finite intersection property in A .

Conversely,let μ be a countable r -cover of A and $r \in pr(L)$ .If none of the finite subfamily ν of μ

is r^+ -cover of A ,then every $t \in \alpha^*(r)$ there is an $x \in \varepsilon_{r'}(A)$ such that $C(x) \le t$ for each $C \in \nu$ .And so

$(\vee\nu)(x) \le t$,equivalently, $(\wedge\nu')(x) \ge t'$ .This shows that subfamily of GL-closed sets μ' having an

r^+ -finite intersection property in A .Hence there is an $x \in \varepsilon_{r'}(A)$ such that $(\wedge\mu')(x) \ge r'$,i.e. $(\vee\mu)(x)$

$\le r$ .This implies that μ is not a countable r -cover of A ,a contradiction.By the Theorem2.4, A is
generalized countably L-compact.

Theorem2.7. Let (L^X, δ) be a GL-ts and $A, B \in L^X$ .If A is generalized countably L-compact and

$B \in \delta'$,then $A \wedge B$ is generalized countably L-compact.

Proof. $\quad$ Let $\quad \phi \subset \delta' \quad$ be $\quad$ a $\quad$ countable $\quad \alpha \quad$ -C-RF of $A \wedge B$ $(\alpha \in M(L))$ .Then $\phi_1 = \phi \cup \{B\}$ is a countable α -C-RF of A .In fact,for each $x_\alpha \in B$ then $x_\alpha \in A \wedge B$ .Hence,there is $P \in \phi \subset \phi_1$ such that $P \in \eta(x_\alpha)$ .If $x_\alpha \notin B$,then $B \in \phi$ and $B \in \eta(x_\alpha)$ .Thus, ϕ_1 is indeed a countable α -C-RF of A .Since A is generalized countably L-compact.there exists an $r \in \beta^*(\alpha)$ and finite subfamily ψ_1 of ϕ_1 such that ψ_1 is an r -C-RF of A .Let $\psi = \psi_1 - \{B\}$,then ψ is a finite subfamily of ϕ ,and ψ is an r -C-RF of $A \wedge B$ .In fact, $x_r \in A \wedge B$,then $x_r \in A$,from the definition of ψ_1 ,there is $P \in \psi_1 \quad$, $\quad$ with $\quad P \in \eta_1(x_r) \quad$ .But $\quad x_r \in B \quad$,so $\quad P \ne B \quad$,and thus $P \in \psi_1 - \{B\} = \psi$ .Hence, $A \wedge B$ is generalized countably L-compact.

Theorem2.8. If A and B are generalized countably L-compact in GL-ts

(L^X, δ),then $A \vee B$ is generalized countably L-compact.

Proof. This is analogous to the proof of the theorem4.1(2) in [3].

Theorem2.9. Let (L^X, δ) and (L^Y, τ) be two GL-ts's, $f^\rightarrow : L^X \rightarrow L^Y$ a GL-continuous mapping and A a generalized countable L-compact set in (L^X, δ) .Then $f^\rightarrow(A)$ is generalized countable L-compact in (L^Y, τ) .

Proof. Let $\phi \subset \tau'$ be a countable α -C-RF of $f^\rightarrow(A)$ and $x_\alpha \in A(\alpha \in M(L))$ .To begin with,let us show that $f^\leftarrow(\phi) = \{f^\leftarrow(P) : P \in \phi\}$ a countable α -C-RF of A .Since $f^\rightarrow$ is GL-continuous and $x_\alpha \in A$, $f^\leftarrow(\phi) \subset \delta'$ and $f^\rightarrow(x_\alpha) = (f^\rightarrow(x))_\alpha \leq f^\rightarrow(A)$ .By ϕ is a countable α -C-RF of $f^\rightarrow(A)$, there is a $P \in \phi$ with $P \in \eta((f^\rightarrow(x))_\alpha)$,i.e. $(f^\rightarrow(x))_\alpha \leq P$,or,equivalently, $P(f^\rightarrow(x)) \geq \alpha$ .By the definition of inverse mapping, $f^\leftarrow(P)(x) = P(f^\rightarrow(x)) \geq \alpha$,hence $x_\alpha \notin f^\leftarrow(P)$,i.e. $f^\leftarrow(P) \in \eta(x_\alpha)$.

Therefore $f^\leftarrow(\phi)$ is a countable α -C-RF of A .

Since A is generalized countably L-compact,there exists an $r \in \beta^*(\alpha)$ and a finite subfamily ψ of ϕ such that $f^\leftarrow(\psi)$ is an r -C-RF of A .Again,by the generalized countable L-compactness of A ,there exists an $r_1 \in \beta^*(r)$ and a finite subset ψ_1 of $f^\leftarrow(\psi)$ such that ψ_1 is an r_1 -C-RF of A .Obviously we can take $\psi_1 = f^\leftarrow(\psi)$.

Now we will show that ψ is an r -C-RF of $f^\rightarrow(A)$ .Let $y_r \leq f^\rightarrow(A)$;by the Lemma 4.10 [8], $r = \sup\{\lambda \in L : \exists x \in f^\leftarrow(y), A(x) \geq \lambda and \lambda \leq r\}$ .Since $r_1 \in \beta^*(r)$, we have $r_1 \in \beta(r)$ and hence there is a $\lambda \in L$ and $x \leq f^\leftarrow(y)$ with $A(x) \geq \lambda$, $\lambda \leq r$,and $\lambda \geq r_1$;thus $x_{r_1} \leq A$ .It follows from $f^\leftarrow(\psi)$ s an r -C-RF of A that there is a $P \in \psi$ with $f^\leftarrow(P) \in \eta(x_{r_1})$,i.e. $f^\leftarrow(P)(x) \geq r_1$ .Hence $P(y) = P(f^\rightarrow(x)) \not\geq r_1$ and therefore certainly $P(y) \geq r$,i.e. $P \in \eta(y_r)$ .Thus $f^\rightarrow(A)$ is generalized countably GL-compact.

Corollary2.10. Let (L^X, δ) be a countable L-compact space and $f : (L^X, \delta) \to (L^Y, \tau)$ a surjective GL-continuous mapping.Then (L^Y, τ) is generalized countably L-compact.

3.Generalized Lindelöf sets

Definition3.1. Let (L^X, δ) be a GL-ts and $A \in L^X$. A is called generalized Lindelöf sets if every α -C-RF ϕ of A has a countable subfamily which is an α^- -C-RF ϕ of A ($\alpha \in M(L)$). (L^X, δ) is called generalized Lindelöf space if 1_X is generalized Lindelöf.

From the Definitions 3.1 and 2.6 we immediately obtain the following results.

Theorem3.2. Let (L^X, δ) be a GL-ts and $A \in L^X$ .Then A is generalized L-compact set if and only if A is generalized Lindelöf and generalized countably L-compact.

Analogous to generalized countable L-compactness,we have the following results.

Theorem3.3. Let (L^X, δ) be a GL-ts and $r \in pr(L)$ .Then A is generalized Lindelöf set if and only if every r -cover μ of A has a countable subfamily ν which is an r^+ -cover of A .

Theorem3.4. Let (L^X, δ) be a GL-ts and $A, B \in L^X$ .If A is generalized Lindelöf set and $B \in \delta'$,then

$A \wedge B$ is generalized Lindelöf.

Theorem3.5. If A and B is generalized Lindelöf sets in GL-ts (L^X, δ) ,then $A \vee B$ is generalized Lindelöf.

Theorem3.6. Let (L^X, δ) and (L^Y, τ) be two GL-ts's, $f^\to : L^X \to L^Y$ a GL-continuous mapping and A a generalized Lindelöf set in (L^X, δ) .Then $f^\to(A)$ is generalized Lindelöf in (L^Y, τ) .

Corollary3.7. Let (L^X, δ) and (L^Y, τ) be a generalized Lindelöf set and $f : (L^X, \delta) \to (L^Y, \tau)$ a surjective GL-continuous mapping.Then (L^Y, τ) is generalized Lindelöf.

Acknowledgments

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