Numerical Algorithm of Two Mixed Finite Element Schemes for A Fourth-Order Diffusion Equation

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Abstract

Two numerical algorithms based on H^1-Galerkin mixed finite element (GMFE) methods are presented and analyzed for a fourth-order diffusion equation. By introducing three variables with two different ways, the two first-order system of four equations are formulated. The semi-discrete optimal error estimates are derived for a numerical scheme and the fully discrete analysis of optimal error results for the other one are made. Moreover, the numerical procedures for two numerical schemes are made to confirm the theoretical results of theorems.

Keywords: Fourth-order diffusion equation; H1- GMFE methods; LBB condition; Error results.

1 Introduction

Fourth-order partial differential equations (PDEs) with high-order spatial derivatives can be found in many scientific fields. So far, many authors have studied some numerical methods for fourth-order PDEs, such as mixed finite element methods [1,4]. Here, we consider a fourth-order diffusion system with initial-boundary value conditions

$$\begin{cases} u_t - u_{xx} - u_{xxt} + u_{xxxx} = f(x,t), & (x,t) \in \Omega \times J, \\ u(x,t) = u_{xx}(x,t) = 0, & (x,t) \in \partial\Omega \times \bar{J}, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1}$$

where $\Omega = (0,1), J = (0,T]$ ($0 < T < \infty$). $f(x,t)$ and $u_0(x)$ are given functions.

H^1- GMFE method, which was proposed by Pani [3] in 1998, has numerically solved many PDEs [4-8]. However, the convergence of H^1-GMFE method for fourth-order diffusion equation (1) has not been studied. In this paper, two numerical schemes based on H^1-GMFE method are shown for fourth-order diffusion equation. Some optimal error estimates are derived in this paper. Finally, some numerical results are provided to verify our theoretical analysis. Throughout this paper, C will denote a generic positive constant which does not depend on parameters h and Δt.

2An H^1- GMFE method for the system (I)

With $q = u_x, v = q_x, \sigma = v_x$ we reformulate problem (1) as the first-order system (I) $u_x = q, v_x = \sigma, v - q_x = 0, u_t - q_{xt} - q_x + \sigma_x = f(x,t)$. (2)

The mixed weak formulation is to find $\{u, v; q, \sigma\} : [0,T] \to H_0^1 \times H^1$ satisfying

$$\begin{cases} (a)\ (u_x, \chi_x) = (q, \chi_x), \forall \chi \in H_0^1, \\ (b)\ (v_x, \omega_x) = (\sigma, \omega_x), \forall \omega \in H_0^1, \\ (c)\ (\sigma, \psi) + (q_x, \psi_x) = 0, \forall \psi \in H^1, \\ (d)\ (q_t, \phi) + (q_{xt}, \phi_x) + (q_x, \phi_x) - (\sigma_x, \phi_x) = -(f, \phi_x), \forall \phi \in H^1. \end{cases} \tag{3}$$

Then the semidiscrete H^1 -GMFE scheme is to look for $\{u_h, v_h; q_h, \sigma_h\} : [0,T] \to V_h \times W_h$ such that , for $V_h \subset H_0^1$ and $W_h \subset H^1$

$$\begin{cases} (a)\ (u_{hx}, \chi_{hx}) = (q_h, \chi_{hx}), \forall \chi_h \in V_h, \\ (b)\ (v_{hx}, \omega_{hx}) = (\sigma_h, \omega_{hx}), \forall \omega_h \in V_h, \\ (c)\ (\sigma_h, \psi_{hx}) + (q_{hx}, \psi_{hx}) = 0, \forall \psi_h \in W_h, \\ (d)\ (q_{ht}, \phi_h) + (q_{hxt}, \phi_{hx}) + (q_{hx}, \phi_{hx}) - (\sigma_{hx}, \phi_{hx}) = -(f, \phi_{hx}), \forall \phi_h \in W_h. \end{cases} \tag{4}$$

For use in the error analysis, we define the elliptic projection $\tilde{u}_h, \tilde{v}_h \in V_h$ by

$$(u_x - \tilde{u}_{hx}, \chi_{hx}) = 0, (v_x - \tilde{v}_{hx}, \omega_{hx}) = 0, \chi_h, \omega_h \in V_h. \tag{5}$$

Further, we also define a Ritz projection $\tilde{q}_h, \tilde{\sigma}_h \in W_h$ of q, σ as the solution of

$$A(q - \tilde{q}_h, \phi_h) = 0, A(\sigma - \tilde{\sigma}_h, \psi_h) = 0, \phi_h, \psi_h \in W_h. \tag{6}$$

where $A(z,w) = (z_x, w_x) + \lambda(z,w)$. Here λ is chosen appropriately so that A is H^1-coercive.

With $\eta = u - \tilde{u}_h, \tau = v - \tilde{v}_h, \rho = q - \tilde{q}_h, \delta = \sigma - \tilde{\sigma}_h$, the following estimates are well known[2]: for $j = 0,1$

$$\|\eta\|_j \le Ch^{k+1-j}\|u\|_{k+1}, \quad \|\tau\|_j \le Ch^{k+1-j}\|v\|_{k+1}, \tag{7}$$

$$\|\rho\|_j \le Ch^{r+1-j}\|q\|_{r+1}, \|\rho_t\|_j \le Ch^{r+1-j}\|q_t\|_{r+1},$$

$$\|\delta\|_j \le Ch^{r+1-j}\|\sigma\|_{r+1}, \|\delta_t\|_j \le Ch^{r+1-j}\|\sigma_t\|_{r+1}. \tag{8}$$

For the convenience, we rewrite $\tilde{u}_h - u_h = \varsigma,\ \tilde{v}_h - v_h = \theta,\ \tilde{q}_h - q_h = \xi,\ \tilde{\sigma}_h - \sigma_h = \gamma.$

From (3)-(6), we then obtain

$$\begin{cases} (a)\ (\varsigma_x, \chi_{hx}) = (\rho, \chi_{hx}) + (\xi, \chi_{hx}), \forall \chi_h \in V_h, \\ (b)\ (\theta_x, \omega_{hx}) = (\delta, \omega_{hx}) + (\gamma, \omega_{hx}), \forall \omega_h \in V_h, \\ (c)\ (\gamma, \psi_h) + (\xi_x, \psi_{hx}) = -(\delta, \psi_h) + \lambda(\rho, \psi_h), \forall \psi_h \in W_h, \\ (d)\ (\xi_t, \phi_h) + (\xi_x, \phi_{hx}) + (\xi_x, \phi_{hx}) + (\xi_{xt}, \phi_{hx}) - (\gamma_x, \phi_{hx}) = -(\rho_t, \phi_h) + \lambda(\rho + \rho_t, \phi_h) - \lambda(\delta, \phi_h), \forall \phi_h \in W_h. \end{cases} \tag{9}$$

Theorem 1 With $q_0 = u_{0x}$, assume that $q_h(0) = \tilde{q}_h(0)$, then

$$\|u - u_h\| + \|v - v_h\| + h(\|u - u_h\|_1 + \|v - v_h\|_1) \le Ch^{\min(k+1, r+1)},$$
$$\|q - q_h\| + \|\sigma - \sigma_h\| + h(\|q - q_h\|_1 + \|\sigma - \sigma_h\|_1) \le Ch^{r+1}.$$

Proof. Choose $\psi_h = \gamma$ in (9c) and $\phi_h = \xi$ in (9d) and add the two equations to get

$$\frac{1}{2}\frac{d}{dt}\left(\|\xi\|^2 + \|\xi_x\|^2\right) + \|\xi\|^2 + \|\gamma\|^2 \le C\left(\|\rho\|^2 + \|\delta\|^2 + \|\rho_t\|^2 + \|\xi\|^2\right) + \frac{1}{2}\|\gamma\|^2. \tag{10}$$

Integrate (10) with respect to time *t* and use Gronwall Lemma to get

$$\|\xi\|^2 + \|\xi_x\|^2 + \int_0^t \left(\|\xi\|^2 + \|\gamma\|^2\right) ds \le C\int_0^t \left(\|\rho\|^2 + \|\delta\|^2 + \|\rho_t\|^2\right) ds. \tag{11}$$

Differentiate (9c) and choose $\psi_h = \gamma$ in (9c) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\gamma\|^2 + (\xi_{xt}, \gamma_x) = -(\delta_t, \gamma) + \lambda(\rho_t, \gamma). \tag{12}$$

Integrate (12) with respect to t and use some important inequality to get

$$\|\gamma\|^2 \le \int_0^t \left(\|\xi_{xt}\|^2 + \|\gamma_x\|^2 + \|\delta_t\|^2 + \|\gamma\|^2 + \|\rho_t\|^2 \right) ds. \tag{13}$$

Choose $\phi_h = \gamma$ in (9d) to obtain

$$\begin{aligned}\|\gamma_x\|^2 &= (\xi_t, \gamma) + (\xi_x, \gamma_x) + (\xi_{xt}, \gamma_x) + \lambda(\rho_t, \gamma) + (\rho_t, \gamma) - \lambda(\rho + \rho_t, \gamma) + \lambda(\delta, \gamma) \\ &\le C\left(\|\xi_t\|^2 + \|\xi_{xt}\|^2 + \|\xi_x\|^2 + \|\gamma\|^2 + \|\rho\|^2 + \|\rho_t\|^2 \right) + \frac{1}{2}\|\gamma_x\|^2.\end{aligned} \tag{14}$$

Choose $\phi_h = \xi_t$ in (9d) to obtain

$$\begin{aligned}\|\xi_t\|^2 + \|\xi_{xt}\|^2 &= -(\xi_x, \xi_{xt}) + (\gamma_x, \xi_{xt}) - (\rho_t, \xi_t) + \lambda(\rho + \rho_t, \xi_t) - \lambda(\delta, \xi_t) \\ &\le C\left(\|\xi_x\|^2 + \|\gamma\|^2 + \|\rho\|^2 + \|\rho_t\|^2 \right) + \frac{1}{2}\left(\|\xi_t\|^2 + \|\xi_{xt}\|^2 \right).\end{aligned} \tag{15}$$

Using (11), we obtain

$$\|\xi_t\|^2 + \|\xi_{xt}\|^2 \le C\left(\|\gamma\|^2 + \|\rho\|^2 + \|\rho_t\|^2 + \int_0^t \left(\|\rho\|^2 + \|\delta\|^2 + \|\rho_t\|^2 \right) ds \right). \tag{16}$$

Combining (11) , (13) , (14) and (16), we use Gronwall lemma to easily obtain

$$\|\gamma\|^2 + \|\gamma_x\|^2 \le C\left(\|\rho\|^2 + \|\rho_t\|^2 + \int_0^t \left(\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 + \|\delta_t\|^2 \right) ds \right). \tag{17}$$

Substitute (17) into (16) to get

$$\|\xi_t\|^2 + \|\xi_{xt}\|^2 \le C\left(\|\rho\|^2 + \|\rho_t\|^2 + \int_0^t \left(\|\rho\|^2 + \|\delta\|^2 + \|\rho_t\|^2 + \|\delta_t\|^2 \right) ds \right). \tag{18}$$

Taking $\chi_h = \varsigma$ in (9a) and $\omega_h = \theta$ in (9b) and using Poincare inequality, we can get

$$\|\varsigma\|^2 \le C\|\varsigma_x\|^2 \le C(\|\rho\|^2 + \|\xi\|^2), \quad \|\theta\|^2 \le C\|\theta_x\|^2 \le C(\|\delta\|^2 + \|\gamma\|^2).$$
(19)

Combining (17), (19), (11), (13), (7)-(8), we apply the triangle inequality to complete the proof.

3 An H1- GMFE method for the system (II)

If our concern is to approximate $q = u_x, v = q_x - u_t - u, \sigma = v_x$ accurately, we rewrite (1) as the first-order system (II)

$$u_x = q, v_x = \sigma, v - q_x + u_t + u = 0, u_t + \sigma_x = f(x,t). \tag{20}$$

The following weak formulation of (20) is to find $\{u, v, q, \sigma\} : [0,T] \to H_0^1 \times H^1$ satisfying

$$\begin{cases} (a)\ (u_x, \chi_x) = (q, \chi_x), \forall \chi \in H_0^1, \\ (b)\ (v_x, \omega_x) = (\sigma, \omega_x), \forall \omega \in H_0^1, \\ (c)\ (\sigma, \psi) + (q_t, \psi) + (q, \psi) + (q_x, \psi_x) = 0, \forall \psi \in H^1, \\ (d)\ (q_t, \phi) - (\sigma_x, \phi_x) = -(f, \phi_x), \forall \phi \in H^1. \end{cases} \tag{21}$$

For the backward Euler procedure, let $\{t_n\}|_0^M$ be a given partition of interval $[0,T]$ with step length $\Delta t = T/M$, $t_n = n\Delta t$, for some positive integer M. For a smooth function ϕ on $[0,T]$. define $\phi^n = \phi(t_n)$ and $\overline{\partial}_t \phi^n = (\phi^n - \phi^{n-1})/\Delta t$. Let U^n, Q^n, V^n and Z^n, respectively, be the approximations of u, q, v and σ at $t = t_n$. The fully scheme is to determine $\{U^n, V^n; Q^n, Z^n\}$ in $V_h \times W_h$ satisfying

$$\begin{cases} (a)\ (U_x^n, \chi_{hx}) = (Q^n, \chi_{hx}), \forall \chi_h \in V_h, \\ (b)\ (V_x^n, \omega_{hx}) = (Z^n, \omega_{hx}), \forall \omega_h \in V_h, \\ (c)\ (Z^n, \psi_h) + (\overline{\partial}_t Q^n, \psi_h) + (Q^n, \psi_h) + (Q_x^n, \psi_{hx}) = 0, \forall \psi_h \in W_h, \\ (d)\ (\overline{\partial}_t Q^n, \phi_h) - k(Z_x^n, \phi_{hx}) = -(f^n, \phi_{hx}), \forall \phi_h \in W_h. \end{cases} \tag{22}$$

For error estimates, we now rewrite the errors as

$$u(t_n) - U^n = (u(t_n) - \tilde{u}_h(t_n)) + (\tilde{u}_h(t_n) - U^n) = \eta^n + \varsigma^n,$$
$$v(t_n) - V^n = (v(t_n) - \tilde{v}_h(t_n)) + (\tilde{v}_h(t_n) - V^n) = \tau^n + \theta^n,$$

$$q(t_n) - Q^n = (q(t_n) - \tilde{q}_h(t_n)) + (\tilde{q}_h(t_n) - Q^n) = \rho^n + \xi^n,$$
$$\sigma(t_n) - Z^n = (\sigma(t_n) - \tilde{\sigma}_h(t_n)) + (\tilde{\sigma}_h(t_n) - Z^n) = \delta^n + \gamma^n.$$

Using (21) and (22), we then obtain

$$\begin{cases} (a)\ (\varsigma_x,\chi_{hx}) = (\rho^n,\chi_{hx}) + (\xi^n,\chi_{hx}), \forall \chi_h \in V_h, \\ (b)\ (\theta_x^n,\omega_{hx}) = (\delta^n,\omega_{hx}) + (\gamma^n,\omega_{hx}), \forall \omega_h \in V_h, \\ (c)\ (\gamma^n,\psi_h) + (\bar{\partial}_t\xi^n,\psi_h) + (\xi^n,\psi_h) + A(\xi^n,\psi_h) \\ \quad = -(\delta^n,\psi_h) - (\bar{\partial}_t\rho^n + \varepsilon^n,\psi_h) - (\rho^n,\psi_h) + \lambda(\xi^n + \rho^n,\psi_h), \forall \psi_h \in W_h, \\ (d)\ (\bar{\partial}_t\xi^n,\phi_h) - A(\gamma^n,\phi_h) = -(\bar{\partial}_t\rho^n + \varepsilon^n,\phi_h) - \lambda(\delta^n + \gamma^n,\phi_h), \forall \phi_h \in W_h. \end{cases} \tag{23}$$

where $\varepsilon^n = q_t(t_n) - \bar{\partial}_t q(t_n)$.

Theorem 3 Assume that $Z^0 = \tilde{q}_h(0)$ with $q_0 = u_{x0}$. Then there exists a positive constant C

independent of h and Δt □ such that for $0 < \Delta t \le \Delta t_0$ and $J = 0,1,\cdots,M$

$$\left\|q^J - Q^J\right\|_j + \left\|\sigma^J - Z^J\right\|_j \le C\left(h^{r+1-j} + \Delta t\right),$$
$$\left\|u^J - U^J\right\|_j + \left\|v^J - V^J\right\|_j \le C\left(h^{min(k+1-j,r+1)} + \Delta t\right),\ j = 0,1.$$

Proof. Choosing $\chi_h = \varsigma^n, \omega_h = \theta^n$ in (23a) and (23b), respectively, we have

$$\left\|\varsigma^n\right\|^2 \le C\left\|\varsigma_x^n\right\|^2 \le C\left(\left\|\rho^n\right\|^2 + \left\|\xi^n\right\|^2\right), \quad \left\|\theta^n\right\|^2 \le C\left\|\theta_x^n\right\|^2 \le C\left(\left\|\delta^n\right\|^2 + \left\|\gamma^n\right\|^2\right). \tag{24}$$

Set $\omega_h = \bar{\partial}_t\xi^n$ in (23c) to get

$$\frac{1}{2}\bar{\partial}_t\left\|\xi^n\right\|_1^2 + \left\|\bar{\partial}_t\xi^n\right\|^2 \le \left\|\bar{\partial}_t\xi^n\right\|^2 + \left(\xi^n,\bar{\partial}_t\xi^n\right) + \left(\xi_x^n,\bar{\partial}_t\xi_x^n\right)$$

$$\begin{aligned} &= -\left(\delta^n,\bar{\partial}_t\xi^n\right) - \left(\bar{\partial}_t\rho^n + \varepsilon^n,\bar{\partial}_t\xi^n\right) - \left(\rho^n,\bar{\partial}_t\xi^n\right) + \lambda\left(\rho^n,\bar{\partial}_t\xi^n\right) \\ &\le \left(\left\|\delta^n\right\|^2 + \left\|\bar{\partial}_t\rho^n\right\| + \left\|\varepsilon^n\right\|^2 + \left\|\rho^n\right\|^2\right) + \frac{1}{2}\left\|\bar{\partial}_t\xi^n\right\|^2. \end{aligned} \tag{25}$$

Summing from $n = 1$ to J, we obtain

$$\left\|\xi^J\right\|_1^2 + 2\Delta t\sum_{n=1}^{J}\left\|\bar{\partial}_t\xi^n\right\|^2 \le 2\Delta t\sum_{n=1}^{J}\left(\left\|\delta^n\right\|^2 + \left\|\bar{\partial}_t\rho^n\right\|^2 + \left\|\varepsilon^n\right\|^2 + \left\|\rho^n\right\|^2\right). \tag{26}$$

Taking $\psi_h = \xi^n$ in (23c), we get

$$\left\|\gamma^n\right\|^2+\left(\bar{\partial}_t \xi^n,\gamma^n\right)+\left(\xi^n,\gamma^n\right)+\left(\xi_x^n,\gamma_x^n\right)=-\left(\delta^n,\gamma^n\right)-\left(\bar{\partial}_t\rho^n+\varepsilon^n,\gamma^n\right)-\left(\rho^n,\gamma^n\right)+\lambda\left(\rho^n,\gamma^n\right). \tag{27}$$

Take $\psi_h=\xi^n$ in (23d) to get

$$-\left(\bar{\partial}_t \xi^n,\gamma^n\right)+\left(\gamma_x^n,\gamma_x^n\right)=-\left(\bar{\partial}_t\rho^n+\varepsilon^n,\gamma^n\right)-\lambda\left(\delta^n,\gamma^n\right). \tag{28}$$

Adding (27) and (28) and using (26), we get

$$\left\|\gamma^J\right\|_1^2\le C\left(\left\|\delta^J\right\|^2+\left\|\bar{\partial}_t\rho^J\right\|^2+\left\|\varepsilon^J\right\|^2+\left\|\rho^J\right\|^2+2\Delta t\sum_{n=1}^{J}\left(\left\|\delta^n\right\|^2+\left\|\bar{\partial}_t\rho^n\right\|^2+\left\|\varepsilon^n\right\|^2+\left\|\rho^n\right\|^2\right)\right). \tag{29}$$

Noting that

$$\left\|\bar{\partial}_t\rho^J\right\|^2\le Ch^{2(r+1)}\frac{1}{\Delta t}\int_{t_{J-1}}^{t_J}\left\|q_t(s)\right\|_{r+1}^2 ds\le Ch^{2(r+1)}\left\|q_t\right\|_{L^\infty\left(H^{r+1}\right)}^2,\ \left\|\varepsilon^n\right\|^2\le C\Delta t\int_{t_{n-1}}^{t_n}\left\|q_{tt}(s)\right\|^2 ds\le C(\Delta t)^2\left\|q_{tt}\right\|_{L^\infty\left(L^2\right)}^2,$$
$$\Delta t\sum_{n=1}^{J}\left(\left\|\bar{\partial}_t\rho^n\right\|^2+\left\|\varepsilon^n\right\|^2\right)\le C\left[h^{2(r+1)}\int_{t_0}^{t_J}\left\|q_t(s)\right\|_{r+1}^2 ds+(\Delta t)^2\int_{t_0}^{t_J}\left\|q_{tt}(s)\right\|^2 ds\right]. \tag{30}$$

Combining (30), (29), (24), (7), (8) and using the triangle inequality, we complete the proof.

4 Some numerical results

In (1), when taking $f(x,t)=\left(\pi^4-1\right)e^{-t}\sin(\pi x)$, $u_0(x)=\sin(\pi x)$ and $T=1$, we easily verify that the exact solution is $u(x,t)=e^{-t}\,sin(\pi x)$. The corresponding basis functions are piecewise linear functions. The numerical results shown in Tables 1-2 illustrate the feasibility of our method .

Table 1: $L^\infty\left(L^2\right)$-errors and convergence rate for scheme (I)

$\Delta t=8h$	$\left\|u-u_h\right\|_{L^\infty(L^2)}$	Rate	$\left\|q-q_h\right\|_{L^\infty(L^2)}$	Rate	$\left\|v-v_h\right\|_{L^\infty(L^2)}$	Rate	$\left\|\sigma-\sigma_h\right\|_{L^\infty(L^2)}$	Rate
1/80	2.42E-03	\	7.91E-03	\	1.90E-03	\	8.08E-03	\
1/160	1.29E-03	0.91	4.13E-03	0.94	1.16E-03	0.72	4.18E-03	0.95
1/320	6.67E-04	0.95	2.12E-03	0.97	6.32E-04	0.87	2.13E-03	0.97
1/640	3.40E-04	0.97	1.07E-03	0.98	3.31E-04	0.93	1.07E-03	0.99

Table 2: $L^{\infty}\left(L^2\right)$-errors and convergence rate for scheme (II)

$\Delta t = 4h$	$\lVert u-u_h \rVert_{L^{\infty}\left(L^2\right)}$	Rate	$\lVert q-q_h \rVert_{L^{\infty}\left(L^2\right)}$	Rate	$\lVert v-v_h \rVert_{L^{\infty}\left(L^2\right)}$	Rate	$\lVert \sigma-\sigma_h \rVert_{L^{\infty}\left(L^2\right)}$	Rate
1/80	2.04E-03	\	7.45E-03	\	2.28E-02	\	8.21E-02	\
1/160	1.19E-03	0.77	4.01E-03	0.89	1.25E-02	0.87	4.17E-02	0.98
1/320	6.43E-04	0.89	2.09E-03	0.94	6.51E-03	0.94	2.11E-02	0.98
1/640	3.33E-04	0.95	1.06E-03	0.97	3.33E-03	0.97	1.06E-02	0.99

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