Journal of Nonlinear Mathematical Physics Volume 15, Number 4 (2008), 353–364 Article

# Contiguity relations for discrete and ultradiscrete Painlevé equations

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*Received May 19, 2008; Accepted August 21, 2008*

**Abstract**

We show that the solutions of ultradiscrete Painlevé equations satisfy contiguity relations just as their continuous and discrete counterparts. Our starting point are the relations for $q$-discrete Painlevé equations which we then proceed to ultradiscretise. In this paper we obtain results for the one-parameter $q$-$\mathrm{P_{III}}$, the symmetric $q$-$\mathrm{P_{IV}}$ and the $q$-$\mathrm{P_{VI}}$. These results show that there exists a perfect parallel between the properties of continuous, discrete and ultradiscrete Painlevé equations.

## 1 Introduction

Painlevé equations have occupied a unique position among ordinary differential equations for almost a century. As a matter of fact, these systems are the simplest nontrivially integrable systems, as Kruskal [1] has repeatedly pointed out in his lectures. Anything simpler than Painlevé equations is integrable through elementary methods, usually linearisation. This is for instance the case of the Riccati equations, which can be linearised through a simple Cole-Hopf transformation, while the integration of Painlevé equations necessitates inverse scattering methods [2]. Anything more complicated than Painlevé equations is simply not integrable (although for higher-order ordinary differential equations the question of the existence of transcendents higher than those introduced by the Painlevé equations is more or less open [3].)

This unique position of the Painlevé equations has been challenged in the last decade (and a half) by the discovery of the discrete Painlevé equations [4]. The latter are integrable nonautonomous mappings, which, at the continuous limit, go over to the (continuous) Painlevé equations. Soon after their discovery it became clear that the discrete Painlevé equations are much more general than the continuous ones. While they do contain them as continuous limits, they go beyond them, since the classification [5] of all second order integrable mappings of discrete Painlevé type revealed that one can have systems with up to 7 parameters, while the continuous Painlevé equations have only up to 4 parameters. Moreover, as we have explained in [6], an infinite number of discrete Painlevé equations exist already at second order. Indeed, based on the geometrical description of discrete Painlevé equations, where an equation is obtained from a periodically repeated nonclosed pattern in the appropriate space of some affine Weyl group, it is easy to conclude that *any* such pattern would lead to a discrete Painlevé equation. Finally, the discrete Painlevé equations come in three flavours. One has difference equations, $q$-difference equations, and those dubbed "elliptic" [5,7] where the independent variable and the parameters enter through the argument of elliptic functions. However, the fact that the discrete Painlevé equations have continuous Painlevé equations as continuous limits, by itself, would not justify their designation as "Painlevé". What is more important is that they share with their continuous counterparts a slew of special, integrability-related, properties.

Indeed, a perfect parallel does exist between these properties for continuous and discrete systems. An obviously incomplete list of such properties is the existence of degeneration cascades, Lax pairs, Miura and auto-Bäcklund transformations, special solutions of various kinds and contiguity relations for solutions [8].

As we have argued, on several occasions, discrete systems are the most fundamental entities. Following this logic and the arguments presented above, the discrete Painlevé equations are more fundamental than the continuous ones. Thus it was no surprise that another variant of Painlevé equations emerged as a special limit of the discrete Painlevé equations. We are referring here to the ultradiscrete Painlevé equations [9] which are generalised cellular automata obtained from the discrete Painlevé equations through the ultradiscretisation procedure [10]. The latter is by now well- established and we do not need to recall it in particular detail. One starts by an ansatz for the various variables of the form $a = e^{A/\epsilon}$, (which assumes that $a$ takes only positive values) and then takes the limit $\epsilon \to 0^+$. The identity $\lim_{\epsilon\to 0^+} \log(e^{A/\epsilon} + e^{B/\epsilon}) = \max(A, B)$ plays a capital role in the ultradiscretisation procedure.

Let us give here an example of a discrete Painlevé equation and its two limits. We start from the $q$-discrete Painlevé equation

$$x_{n+1}x_{n-1} = z_n x_n + \sigma \tag{1.1}$$

where $z_n = z_0\lambda^n$ and $\sigma = \pm 1$. The two different choices for the sign of $\sigma$ will allow us to obtain two different limits. For the continuous limit, we put $x = 1 + \epsilon 2w$, $z_n = 2 + \epsilon 4t$ (where $t = n\epsilon$, with $\lambda = 1 + \epsilon 5/2$) and take $\sigma = -1$. At the limit $\epsilon \to 0$, we find

$$w'' = -w2 + t \tag{1.2}$$

i.e. the continuous $P_I$ (albeit in noncanonical form). This is the reason why (1.1) is usually referred to as $q$-$P_I$.

For the ultradiscrete limit we put $x = e^{X/\epsilon}$, $\lambda = e^{1/\epsilon}$ and take $\sigma = +1$. At the limit $\epsilon \to 0$, we obtain

$$X_{n+1} + X_{n-1} = \max(X_n + n, 0) \tag{1.3}$$

This is an ultradiscrete form of $P_I$. We remark readily that (1.3) defines a generalised cellular automaton. Indeed, if the initial values of $X$ are integer, the evolution defined by (1.3) will lead only to integer values for $X$.

The ultradiscrete Painlevé equations do justice to their name since they possess the properties which characterise the discrete and continuous Painlevé equations. In particular, they can be organized in degenerescence cascades, they possess Lax pairs (although it is far from clear how one can use these to solve the spectral problem in this case), they have families of special solutions and there exist interrelations in the form of Miura and auto-Bäcklund transformations [11]. For the latter however very few results exist to date, and in particular no attempt has been made to establish contiguity relations for the solutions of ultradiscrete Painlevé equations as has been done for their discrete and continuous counterparts.

The aim of this paper is precisely to remedy this omission. We shall show, through specific examples, that the solutions of ultradiscrete Painlevé equations do indeed possess contiguity relations, thus establishing a perfect parallel between all three species of Painlevé equations. The way to do this is to start from some $q$-discrete Painlevé equation, obtain the contiguity relations of its solutions and then proceed to ultradiscretise the whole procedure.

## 2 Contiguity relations for continuous and discrete Painlevé equations

Before proceeding to the construction of contiguity relations for the solutions of ultradiscrete equations we shall illustrate the situation for continuous and discrete Painlevé equations through a specific example.

We start from the continuous $\mathrm{P_{III}}$ equation:

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(\alpha w2 + \beta) + \gamma w3 + \frac{\delta}{w} \tag{2.1}$$

Assuming $\gamma \neq 0$, $\delta \neq 0$ one can use scaling of both $w$ and $t$ to get $\gamma = 1$, $\delta = -1$. We have the following relations [12]:

$$w(-\alpha, -\beta) = -w(\alpha, \beta) \tag{2.2}$$

$$w(-\beta, -\alpha) = w^{-1}(\alpha, \beta) \tag{2.3}$$

$$w(-\beta - 2, -\alpha - 2) = w(\alpha, \beta) \left( 1 + \frac{2 + \alpha + \beta}{t(\frac{w'}{w} + w + \frac{1}{w}) - 1 - \beta} \right) \tag{2.4}$$

The last relation is precisely the auto-Bäcklund/Schlesinger transformation of $\mathrm{P_{III}}$. We assume further that $\alpha \neq \beta$. Using the relations (2.2-4), starting from $w(-\beta, -\alpha)$ which leads to $w(\alpha-2, \beta-2)$, we can eliminate $w'$ and obtain a relation between $w(\alpha-2, \beta-2)$, $w(\alpha, \beta)$ and $w(\alpha+2, \beta+2)$, i.e. a one-dimensional 3-point mapping on the $(\alpha, \beta)$-plane. We introduce the independent variable $z = (\alpha + \beta + 2)/4$ (so as to have $\Delta z = 1$) and two parameters $\kappa$, $\mu$ through $\mu = (\beta - \alpha - 2)/4, \kappa = -it/2$. We choose $x = i/w$ as the mapping variable and find:

$$\frac{z_n}{x_{n+1}x_n + 1} + \frac{z_{n-1}}{x_n x_{n-1} + 1} = \kappa(-x_n + \frac{1}{x_n}) + z_n + \mu \tag{2.5}$$

Equation (2.5) is a contiguity relation for the solutions of $P_{III}$. Interpreted as a mapping it is what we have dubbed the "alternate" discrete Painlevé II equation. As shown in [13], the continuous limit of (2.5) is indeed $P_{II}$ (although the qualifier "alternate" is a very unfortunate one).

Since (2.5) is a (discrete) Painlevé equation it is natural to expect the same procedure as for the continuous ones (Schlesinger transformation and contiguity) to hold true. Thus the question is whether we can derive further discrete equations by considering the evolution along the parameters, induced by the discrete Schlesingers. We examine once again the alternate d-$P_{II}$. The Schlesinger transform of alt-d-$P_{II}$ was obtained in [13] and has the form:

$$x_n(\mu-1) = \frac{1}{x_n} + \frac{\mu(1+x_nx_{n-1})}{\kappa(1+x_nx_{n-1}) - z_{n-1}x_n} \tag{2.6}$$

where $x_n(\mu-1)$ satisfies the alternate d-$P_{II}$ equation with parameter $\mu-1$ and $x_n$ without argument is the solution of (2.5) for parameter $\mu$. Similarly we have the solution for parameter $\mu+1$:

$$x_n(\mu+1) = \left(x_n - \frac{(\mu+1)(1+x_nx_{n-1})}{\kappa(1+x_nx_{n-1}) - z_{n-1}x_{n-1}}\right)^{-1} \tag{2.7}$$

The final step consists in eliminating $x_{n-1}$ between (2.6) and (2.7). We obtain thus the dual equation of (2.5), i.e. the equation where the parameter $\mu$ is now the independent variable:

$$\frac{\mu+1}{x_\mu x_{\mu+1}-1} + \frac{\mu}{x_\mu x_{\mu-1}-1} = \kappa(x_\mu + \frac{1}{x_\mu}) - \mu - z \tag{2.8}$$

Here we have dropped the index $n$, and introduced the index $\mu$ with obvious notations. The quantity $z \equiv z_n$ is just a parameter. We remark that (2.8) is the alternate d-$P_{II}$ itself (up to a rescaling of the variable $x$ and of the parameters by a factor of $i$).

This is a most remarkable result. The discrete equation we obtain is the same whether we consider it along the independent variable $n$ or along the parameter $\mu$. This is the property of self-duality and we have shown that it is a general feature of discrete, difference equations which are contiguity relations of the solutions of the continuous Painlevé equations. As a matter of fact the property of self-duality holds true for all difference Painlevé equations [14]. On the contrary, there exist $q$-Painlevé equations for which the self-duality is violated [5]. Still, for the purpose of the present paper this is even more interesting: in the non-self-dual cases one can derive the contiguity relations of $q$-Painlevé equations and obtain new $q$-discrete Painlevé equations.

## 3 Contiguity of the one-parameter $q$-$P_{III}$ and its ultradiscretisation

As explained in the introduction, in order to obtain the contiguity relations of the solutions of some ultradiscrete equation we start from a $q$-discrete equation which produces the ultradiscrete one at the appropriate limit. In this section we shall examine the one-parameter $q$-$P_{III}$ we introduced in [15].

Our starting point is the mapping

$$x_n x_{n+1} = \frac{1 + y_n/z_n}{y_n(1 + y_n/d)} \tag{3.1$a$}$$

$$y_n y_{n-1} = d\frac{x_n + 1}{x_n 2} \tag{3.1$b$}$$

where $z$ is the independent variable $z_n = z_0\lambda^n$. We proceed by obtaining the Schlesinger transformations, which correspond to the changes of the parameter $d$. We shall denote these changes by an evolution in the direction $m$ i.e. $d_{m+1} = \lambda d_m$. We first introduce an auxiliary variable $v$ through the relations

$$x_{n+1}yv = xyv_{m+1} = 1 \tag{3.2}$$

where we have omitted the indexes $n, m$ when they are not up-shifted. Using (3.1) and (3.2) we can now establish the equations for the evolution along the $m$ direction:

$$v_m v_{m+1} = \frac{1 + y_m/d_m}{y_m(1 + y_m/z)} \tag{3.3$a$}$$

$$y_m y_{m-1} = z\frac{v_m + 1}{v_m 2} \tag{3.3$b$}$$

where $z$ does not vary in the direction $m$. We remark that this equation has the same form as (3.1), provided one replaces $x$ by $v$ and interchanges $z$ and $d$. Thus the one- parameter $q$-$\mathrm{P_{III}}$ is self-dual. (This is a remarkable result since the geometry of this equation is described by the affine Weyl group $(A_1 + A_1)^{(1)}$, [16], which does not have a structure leading naturally to self-dual equations. Still, the one-parameter $q$-$\mathrm{P_{III}}$ is defined by an evolution in a direction which results in a dual equation of the same form).

However, equation (3.3) is not very nice as a contiguity relation of the solutions of (3.1) since it involves the auxiliary variable $v$. It is more interesting to write the dual equation in terms of the variables $x$ and $y$. The simplest relation we can find is the one relating $x_{m-1}$ and $y_{m-1}$ to $x_{n+1}$ and $y_n$. Using (3.1-3) we find

$$x_{m-1} = \frac{y_n}{z(1 + y_n x_{n+1})} \tag{3.4$a$}$$

$$y_{m-1} = z x_{n+1}(1 + y_n x_{n+1}) \tag{3.4$b$}$$

These relations can be easily inverted. We obtain

$$x_{n+1} = \frac{y_{m-1}}{z(1 + x_{m-1}y_{m-1})} \tag{3.5$a$}$$

$$y_n = z x_{m-1}(1 + x_{m-1}y_{m-1}) \tag{3.5$b$}$$

We can of course introduce relations with a slightly different staggering, for instance $x_{m-1}$ and $y_{m-1}$ in terms of $x_n$ and $y_n$ (solving for $x_{n+1}$ from (3.1a)) but as they look more complicated than (3.5) we prefer not to give them here.

Ultradiscretising the one-parameter $q$-$\mathrm{P_{III}}$ and its contiguities is straightforward. We introduce the ansatz $x = e^{X/\epsilon}$, $y = e^{Y/\epsilon}$, $v = e^{V/\epsilon}$, $\lambda = e^{1/\epsilon}$ and take the limit of the logarithm of the relations obtained above when $\epsilon \to 0$. We find for the one- parameter $q$-$\mathrm{P_{III}}$ the ultradiscrete form

$$X_n + X_{n+1} = \max(0, Y_n - n) - Y_n - \max(0, Y_n - m) \tag{3.6a}$$

$$Y_n + Y_{n-1} = m + \max(0, X_n) - 2X_n \tag{3.6a}$$

where we have used explicitly the fact that $z_n = z_0\lambda^n$, $d_m = d_0\lambda^m$ and absorbed the effect of $z_0, d_0$ in the definition of $n, m$. The relations involving the auxiliary variable are particularly simple

$$X_{n+1} + Y + V = X + Y + V_{m+1} = 0 \tag{3.7}$$

We can now write the ultradiscrete contiguity relation in terms of $V$. We find

$$V_m + V_{m+1} = \max(0, Y_m - m) - Y_m - \max(0, Y_m - n) \tag{3.8a}$$

$$Y_m + Y_{m-1} = n + \max(0, V_m) - 2V_m \tag{3.8b}$$

Finally we give the dual evolution in terms of the variables $X$ and $Y$. We find

$$X_{m-1} = Y_n - n - \max(0, Y_n + X_{n+1}) \tag{3.9a}$$

$$Y_{m-1} = n + X_{n+1} + \max(0, Y_n + X_{n+1}) \tag{3.9b}$$

or, ultradiscretising (3.5),

$$X_{n+1} = Y_{m-1} - n - \max(0, Y_{m-1} + X_{m-1}) \tag{3.10a}$$

$$Y_n = n + X_{m-1} + \max(0, Y_{m-1} + X_{m-1}) \tag{3.10b}$$

Again the dual equation is most simple in terms of the auxiliary variable $V$ but one can indeed write contiguity relations is terms of the variable $X$, $Y$, which are not more complicated if one chooses the most convenient staggering. It is straightforward to verify that $\{X_{m+1}, Y_{m+1}\}$ satisfy indeed equation (3.6) for parameter $m + 1$.

# 4 Contiguities of the symmetric $q$-$\mathrm{P_{IV}}$ and its ultradiscretisation

The symmetric form of q-$\mathrm{P_{IV}}$ was first presented by Kajiwara, Noumi and Yamada in [17] (referred to as KNY in what follows). It has the form

$$\begin{aligned}
\overline{f}_0 &= a_0a_1f_1\frac{1 + a_2f_2 + a_2a_0f_2f_0}{1 + a_0f_0 + a_0a_1f_0f_1}\\
\overline{f}_1 &= a_1a_2f_2\frac{1 + a_0f_0 + a_0a_1f_0f_1}{1 + a_1f_1 + a_1a_2f_1f_2}\\
\overline{f}_2 &= a_2a_0f_0\frac{1 + a_1f_1 + a_1a_2f_1f_2}{1 + a_2f_2 + a_2a_0f_2f_0}
\end{aligned} \tag{4.1}$$

where the "bar" denotes the evolution along the independent variable. Here the $f_i$ are the dependent variables while the $a_i$ are parameters. The reversed evolution is given by

$$\begin{aligned}\underline{f}_0 &= \frac{f_2}{a_0 a_1}\frac{a_0 a_1 + a_0 f_1 + f_0 f_1}{a_2 a_0 + a_2 f_0 + f_2 f_0}\\ \underline{f}_1 &= \frac{f_0}{a_1 a_2}\frac{a_1 a_2 + a_1 f_2 + f_1 f_2}{a_0 a_1 + a_0 f_1 + f_0 f_1}\\ \underline{f}_2 &= \frac{f_1}{a_2 a_0}\frac{a_2 a_0 + a_2 f_0 + f_2 f_0}{a_1 a_2 + a_1 f_2 + f_1 f_2}\end{aligned} \tag{4.2}$$

Following KNY we introduce a constant $\lambda$ setting $a_0 a_1 a_2 = \lambda$. The independent variable is introduced through the product $f_0 f_1 f_2$. As a matter of fact we have $f_0 f_1 f_2 = \lambda z 2$ where $\bar{z} = \lambda z$. The geometry of this equation is described by the affine Weyl group $(A_2 + A_1)^{(1)}$.

The nice symmetric form of KNY notwithstanding, in what follows we shall make a particular choice of variables, thus breaking the symmetry, in order to illustrate one particular Schlesinger transformation. It goes without saying that any choice of two variables out of three is possible. Below we shall work with $x \equiv f_0$ and $y \equiv f_1$ (and put $\alpha = a_0$, $\beta = a_1$). First we eliminate $f_2$ and $a_2$ from equations (4.1) and (4.2) and obtain the following system for $x$ and $y$

$$x_n x_{n+1} = \frac{\alpha\beta x_n y_n + z_{n+1}^2(1 + \alpha x_n)}{1 + \alpha x_n + \alpha\beta x_n y_n} \tag{4.3$a$}$$

$$y_n y_{n+1} = z_{n+1}^2 \frac{1 + \alpha x_n + \alpha\beta x_n y_n}{z_{n+1}^2 + \alpha x_n + \alpha\beta x_n y_n} \tag{4.3$b$}$$

where the index $n$ is associated with the independent variable, i.e. $z_n = z_0 q^n$. We can complement these equations by the reverse evolution

$$x_{n-1} x_n = z_n 2 \frac{\alpha\beta + \alpha y_n + x_n y_n}{\alpha\beta z_n 2 + \alpha y_n + x_n y_n} \tag{4.4$a$}$$

$$y_{n-1} y_n = \frac{x_n y_n + \alpha z_n 2(y_n + \beta)}{\alpha\beta + \alpha y_n + x_n y_n} \tag{4.4$b$}$$

Next we introduce a Schlesinger transformation $S$ of the solutions of (4.3) following the procedure of KNY. It corresponds to an evolution along the parameters $a_0$ and $a_1$: $S(a_0) = \lambda a_0$ and $S(a_1) = a_1/\lambda$ (and obviously $S(a_2) = a_2$). We introduce another index $m$ in order to denote this evolution. We have $\alpha_{m+1} = \lambda\alpha_m$ and $\beta_{m+1} = \beta_m/\lambda$. We can now rewrite the results of KNY as

$$y_m y_{m+1} = \lambda z 2 \frac{1 + \alpha_m x_m}{x_m(\alpha_m + x_m)} \tag{4.5$a$}$$

and

$$x_{m-1} x_m = \lambda z 2 \frac{\beta_m + y_m}{y_m(1 + \beta_m y_m)} \tag{4.5$b$}$$

where $z$ is now a parameter. System (4.5) is the dual equation of (4.3) or, equivalently, the contiguity relation of its solutions. Equation (4.5) was first obtained in [18] and is in fact a $q$- discrete form of $\mathrm{P_{III}}$. The geometry of this $q$ Painlevé equation is, of course, described by the affine Weyl group $(A_2 + A_1)^{(1)}$ and thus in general this equation is non-self-dual. However just as we defined (4.5) by keeping $a_2$ constant we can define Schlesinger evolutions by keeping $a_0$ or $a_1$ constant. These equations are dual not only of (4.3) but also of (4.5). Since they have the same form as the latter the $q$-$\mathrm{P_{III}}$ equation is self-dual when one is limited to evolutions "within the $A_2^{(1)}$ plane". The non-self-dual character is manifested when one considers evolution (4.3) "along the $A_1^{(1)}$ direction, orthogonal to the $A_2^{(1)}$ plane".

We turn now to the ultradiscretisation of $q$-$\mathrm{P_{IV}}$ and its continguity relation. We introduce the ansatz $x = e^{X/\epsilon}$, $y = e^{Y/\epsilon}$, $\lambda = e^{1/\epsilon}$, $\alpha = e^{A/\epsilon}$, $\beta = e^{B/\epsilon}$ and take the limit of the relations obtained above when $\epsilon \to 0$. The ultradiscrete equivalent of $q$-$\mathrm{P_{IV}}$ is

$$X_n + X_{n+1} = \max(A + B + X_n + Y_n, 2(n+1), 2(n+1) + A + X_n)) \\ - \max(0, A + X_n, A + B + X_n + Y_n) \tag{4.6a}$$

$$Y_n + Y_{n+1} = 2(n+1) + \max(0, A + X_n, A + B + X_n + Y_n) \\ - \max(2(n+1), A + X_n, A + B + X_n + Y_n) \tag{4.6b}$$

Similarly, the ultradiscrete limit of (4.4) is

$$X_{n-1} + X_n = 2n + \max(A + B, A + Y_n, X_n + Y_n) \\ - \max(A + B + 2n, A + Y_n, X_n + Y_n) \tag{4.7a}$$

$$Y_{n-1} + Y_n = \max(X_n + Y_n, A + 2n + Y_n, A + 2n + B) \\ - \max(A + B, A + Y_n, X_n + Y_n) \tag{4.7b}$$

The ultradiscretisation of the dual equations of (4.3), which define the contiguity relations of the solutions, can be given in a straightforward way. For the case of the $S$ transformation introduced above we find

$$Y_m + Y_{m+1} = N + \max(0, A_m + X_m) - X_m - \max(A_m, X_m) \tag{4.8a}$$

and

$$X_{m-1} + X_m = N + \max(B_m, Y_m) - Y_m - \max(0, B_m + Y_m) \tag{4.8b}$$

where $N = 2n + 1$ is just a parameter in this case and $A_m = A_0 + m$, $B_m = B_0 - m$. Starting from (4.8) it is straightforward (albeit somewhat lengthy) to show that $\{X_{m+1}, Y_{m+1}\}$ form a solution of (4.6) (and equivalently (4.7)) for parameters $A+1$, $B-1$. Similarly we can show that $\{X_{m-1}, Y_{m-1}\}$ corresponds to parameters $A-1$, $B+1$.

## 5 Contiguity of $q$-$\mathrm{P_{VI}}$ and its ultradiscretisation

In this section we shall examine the $q$-$\mathrm{P_{VI}}$ we introduced in [4,14] and which was shown to be a $q$-discrete form of $\mathrm{P_{VI}}$ by Jimbo and Sakai [19]. Our starting point is the mapping (in a slightly unusual normalisation, permitting an easier transition to the dual equation)

$$x_n x_{n+1} = \frac{(y_n + p z_n)(y_n + \lambda z_n/p)}{(1 + y_n z_n s)(1 + \lambda y_n z_n/s)} \tag{5.1a}$$

$$y_n y_{n-1} = \frac{(x_n + az_n)(x_n + z_n/a)}{(1 + x_n z_n c)(1 + x_n z_n/c)} \tag{5.1b}$$

where $z$ is the independent variable $z_n = z_0\lambda^n$. The Schlesinger transformations correspond to the changes of the parameters $a, c, p, s$. As was done in the previous sections we shall denote these changes by an evolution in the direction $m$. Following our analysis in [14] we introduce an auxiliary variable $v$ through the relations

$$vy = y_{n-1}v_{m+1} = \frac{(x + z/a)\sqrt{a}}{(1 + xz/c)\sqrt{c}} \tag{5.2}$$

or equivalently, because of (5.1)

$$\frac{v}{y_{n-1}} = \frac{v_{m+1}}{y} = \frac{(1 + xzc)\sqrt{a}}{(x + az)\sqrt{c}} \tag{5.3}$$

where we have omitted the indexes $n, m$ when they are not up- or down- shifted. The evolution in the $m$ direction can be easily obtained using (5.2) and (5.3):

$$vv_{m+1} = \frac{(x + 1/(zc))(x + z/a)}{(1 + x/(az))(1 + xz/c)} \tag{5.4}$$

which is similar to (5.1b). Here $z$ is a parameter that does not vary in the direction $m$, and as explained in [14] the parameters $a_m$ and $c_m$ are proportional to $\lambda^{-m}$. Calling $\theta_m = \theta_0\lambda^m$ the variables in the $m$ direction and putting $a_m = r/\theta$, $c_m = 1/(r\theta)$ we can rewrite (5.4) as

$$vv_{m+1} = \frac{(x + \theta r/z)(x + \theta z/r)}{(1 + x\theta/(rz))(1 + x\theta rz)} \tag{5.5a}$$

From [14] the equation that relates $x_{m-1}$, $v$ and $x$ is

$$x_{m-1}x = \frac{(v + \theta p/\lambda)(v + \theta/p)}{(1 + v\theta/s)(1 + v\theta s/\lambda)} \tag{5.5b}$$

We remark that this equation has the same form as (5.1), provided one replaces $y$ by $v$ and considers $\theta$ as the variable and $z$ as a fixed parameter. Thus the $q$-$P_{VI}$ is self-dual (as expected).

Just as in the case of the one-parameter $q$-$P_{III}$, while equation (5.5) is indeed a contiguity relation of the solutions of (5.1), it does not look optimal since it involves the auxiliary variable $v$. Thus it is better to write the dual equation in terms of the variables $x$ and $y$. We find

$$x_m x_{m-1} = \frac{s}{p}\frac{(y_m p\theta(1 + x_m zr\theta) + \lambda(x_m r + z\theta))(y_m\theta(1 + x_m zr\theta) + p(x_m r + z\theta))}{(y_m\lambda(1 + x_m zr\theta) + s\theta(x_m r + z\theta))(y_m s(1 + x_m zr\theta) + \theta(x_m r + z\theta))} \tag{5.6a}$$

and

$$y_m y_{m-1} = \frac{(x_{m-1}\theta + rz)(x_m r + z\theta)}{(x_{m-1}z + r\theta)(1 + x_m rz\theta)} \tag{5.6b}$$

We remark that the equation for $y_{m-1}$ is quite simple, but this is achieved at the price of having both $x_m$ and $x_{m-1}$ appearing in the r.h.s. of the equation.

We now turn to the ultradiscretisation of $q$-P$_{\rm VI}$ and its contiguities. With obvious notations we obtain

$$X_n + X_{n+1} = \max(Y_n, n+P) + \max(Y_n, n+1-P) - \max(0, Y_n+n+S) - \max(0, Y_n+n+1-S) \tag{5.7$a$}$$

$$Y_n + Y_{n-1} = \max(X_n, n+A) + \max(X_n, n-A) - \max(0, X_n+n+C) - \max(0, X_n+n-C) \tag{5.7$b$}$$

The relations involving the auxiliary variable can now be written

$$V + Y = Y_{n-1} + V_{m+1} = \max(X+n-A) - \max(0, X+n-C) + A/2 - C/2 \tag{5.8$a$}$$

and

$$V - Y_{n-1} = V_{m+1} - Y = \max(0, X+n+C) - \max(X, n+A) + A/2 - C/2 \tag{5.8$b$}$$

where again the indexes $n, m$ are omitted when they are not up- or down-shifted. Eliminating $Y$ we can obtain the evolution in the $m$ direction

$$V_m + V_{m+1} = \max(X_n, -Z-C_m) + \max(X_n, Z-A_m) - \max(0, X_n-Z-A_m) - \max(0, X_n+Z-C_m) \tag{5.9}$$

where now $Z \equiv n$ is just a parameter. The parameters $A$ and $C$ on the other hand, do vary along the direction $m$. As in the discrete case it is convenient to introduce the independent variable $m$ explicitly. We have $A_m = R - m$ and $C_m = -R - m$ leading to

$$V_m + V_{m+1} = \max(X_n, -Z+R+m) + \max(X_n, Z-R+m) - \max(0, X_n-Z-R+m) - \max(0, X_n+Z+R+m) \tag{5.10$a$}$$

Similarly the equation relating $X_{m-1}$ to $V_m$ and $X_m$ becomes

$$X_{m-1} + X_m = \max(V_m, m-1+P) + \max(V_m, m-P) - \max(0, V_m+m-S) - \max(0, V_m-1+S) \tag{5.10$b$}$$

The remark of the self-duality of $q$-P$_{\rm VI}$ applies obviously to its ultradiscrete analogue: it suffices to compare the equations (5.7) and (5.10). Finally we present the contiguity relations without reference to the auxiliary variable $V$. We find

$$\begin{aligned} X_{m-1} + X_m = {} & \max(Y_m+P+m+\max(0, X_m+R+Z+m), X_m+R+1, Z+m+1) \\ & + \max(Y_m+P+m+\max(0, X_m+R+Z+m), X_m+R+P, Z+P+m) \\ & - \max(Y_m+1+\max(0, X_m+R+Z+m), X_m+R+S+m, Z+S+2m) \\ & - \max(Y_m+S+\max(0, X_m+R+Z+m), X_m+R+m, Z+2m) + S - P \end{aligned} \tag{5.11$a$}$$

$$Y_m + Y_{m-1} = \max(X_{m-1}+m, R+Z) + \max(X_m+R, Z+m) - \max(X_{m-1}+Z, R+m) - \max(0, X_m+R+Z+m) \tag{5.11$b$}$$

The verification that (5.11) is indeed a contiguity relation of the solutions of (5.7) is straightforward (although it may be quite heavy computationally).

# 6 Conclusions

In this paper we have addressed the question of the existence of contiguity relations for the solutions of ultradiscrete Painlevé equations. The motivation for this was the fact that these very special ultradiscrete systems possess all the richness which characterizes the Painlevé equations, be they continuous or discrete. Just like the latter they are organized in degeneration cascades (obtained though coalescences of their parameters), they have Lax pairs, possess Miura and auto-Bäcklund transformations and for some particular values of their parameters one can present special solutions. It was thus natural to look for the one property of ultradiscrete Painlevé equations which had not been studied till now.

The method used for the derivation of the ultradiscrete contiguity relations was to start from the discrete ones for the appropriate $q$- discrete equation and implement the ultradiscretisation procedure. The equations analysed here were the one-parameter $q$-$\mathrm{P}_{\mathrm{III}}$, the symmetric $q$-$\mathrm{P}_{\mathrm{IV}}$ and the $q$-$\mathrm{P}_{\mathrm{VI}}$. In every case we could obtain the discrete and ultradiscrete contiguity relations. A possible extension of this work would be to apply the present approach to higher $q$-discrete Painlevé equations (where by "higher" we mean equations having more parameters than $q$-$\mathrm{P}_{\mathrm{VI}}$). As shown by Sakai such equations do exist, associated with the affine Weyl groups $\mathrm{E}_6^{(1)}$, $\mathrm{E}_7^{(1)}$ and $\mathrm{E}_8^{(1)}$. Their geometrical study has been performed in [20,21]. Based on the latter one can in principle proceed to the derivation of ultradiscrete contiguities. We intend to return to this problem in some future work of ours.

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