On Archimedean of t-norms

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Abstract—In this paper, we consider the Archimedean properties of t-norms. We summarize some Archimedean properties and characterize some new Archimedean properties of triangular norms on [0,1].

Introduction

Triangular-norms (t-norms) and related operations (such as t-conforms, implications) play an important role e.g. in the fuzzy theory and its applications. Especially the continuous Archimedean t-norms and the continuous t-norms are popular. The latest can be built up with the help of continuous Archimedeans t-norms and the strongest t-norm, the minimum operator. Hence, examining continuous Archimedean t-norms has major importance in the field.

It is well-known from the literature that continuous Archimedean t-norms can be represented by additive generator functions [2].

Triangular norms on [0,1] were introduced in [6] and play an important role in fuzzy set theory (see e.g. [1,7,8] for more details). One of the most important properties that can be satisfied by t-norms on the unit interval is the Archimedean property: continuous t-norms can be fully characterized by means of Archimedean t-norms, the Archimedean property is closely related to additive and multiplicative generators, etc. [1,9,10]. In this paper,we summarize some Archimedean properties and characterize some new Archimedean properties.[11].

Preliminaries

Definition 1. A triangular norm (t-norm for short)is a binary operation T on the unit interval [0,1],i.e.a function $T:[0,1]^2 \to [0,1]$,such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:

(T1) $T(x,y)=T(y,x)$ (commutativity)

(T2) $T(x,T(y,z))=T(T(x,y),z)$ (associativity)

(T3) $T(x,y) \le T(x,z)$,whenever $y \le z$ (monotonicity)

(T4) $T(x,1)=x$ (boundary condition).

Definition 2. The t-norm T is called Archimedean if

(AP) for each $(x,y) \in]0,1[^2$,there is an $n \in \mathbf{N}$ with $x_T^{(n)} < y$.

Definition 3. The t-norm T satisfies the cancellation law if

(CL) $T(x,y)=T(x,z)$ implies x=0 or y=z.

Definition 4. The t-norm T satisfies the conditional cancellation law if

(CCL) $T(x,y)=T(x,z)>0$ implies y=z.

Definition 5. The t-norm T has the limit property if

(lP) for all $x \in]0,1[$: $\lim_{n\to\infty} x_T^{(n)} = 0$

Definition 6. A function F: $[0,1]^2 \to [0,1]$ is continuous if for all convergent sequences $(x_n)_{n\in\mathbf{N}},(y_n)_{n\in\mathbf{N}} \in [0,1]^{\mathrm{N}}$ we have

$$F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} F(x_n, y_n)$$

$$(x*y)*\big((x*y)*(y*x)\big) = 0, \forall x, y \in A.$$

The following examples support that the four conditions in the BCK-algebra definition are independent. Therefore, in order to simplify the BCK-algebra definition we have to propose the equivalent new definition of BCK-algebra.

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The first characterizations to the Archimedean of t-norms

In this paragraph,in the basis of summarizing some Archimedean properties of t-norms,we characterize some new ones and give the proofs.

Theorem 1. A continuous t-norm is Archimedean if and only if it satisfies the CCL.

Proof: A continuous Archimedean t-norm is either nilpotent or strict. Each strict and each nilpotent t-norm fulfills the CCL.Thus necessity and sufficiency are obvious.

Theorem 2. If $\lim_{x \searrow x_0} T(x,x) < x_0$ for each $x_0 \in]0,1[$,then T is Archimedean.

Proof: assume $\lim_{x \searrow x_0} T(x,x) < x_0$ for each $x_0 \in]0,1[$.

Then because of the monotonicity (T3),

we have

$$T(x_0, x_0) \le \lim_{x \searrow x_0} T(x,x) < x_0$$

for all $x_0 \in]0,1[$,and T is Archimedean.

A t-norm T is called strict if it is continuous and strictly monotone.Thus

Theorem 3. If T is strict, then T is Archimedean.

Proof: a strict t-norm is continuous and can have only trivial idempotent elements, Theorem 3 is a special case of Theorem 2.

Therefore we have,

Theorem 4. If T is strictly monotone and continuous then T is Archimedean

Theorem 5. If each $x \in]0,1[$ is a nilpotent element of T,then T is Archimedean.

Proof: if each $x \in]0,1[$ is a nilpotent element of T then T satisfies LP and is Archimedean.

Theorem 6. If T is nilpotent then T is Archimedean.

Proof: each nilpotent t-norm T is isomorphic to T_L and T_L is Archimedean. Obviously T is Archimedean.

Theorem 7. T has the limit property if and only if T is Archimedean.

Theorem 8. T has only trivial idempotent elements and ,whenever

$$\lim_{x \searrow x_0} T(x,x) = x_0$$

for some $x_0 \in]0,1[$,there exists a $y_0 \in]x_0,1[$ such that $T(y_0, y_0) = x_0$ if and only if T is Archimedean.

In fact Theorem 7 is equivalent to Theorem 8. In order Wto simply the proof. we write them as follows:

For a t-non T the following are equivalent:

(i) T is Archimedean.

(ii) T satisfies the limit property (LP).

(iii) T has only trivial idempotent elements and, whenever

$\lim_{x \searrow x_0} T(x,x) = x_0$

for some $x_0 \in]0,1[$,there exists a $y_0 \in]x_0,1[$ such that $T(y_0, y_0) = x_0$ if and only if T is Archimedean.

Proof: In order to show that (i) implies (iii).

Let T be Archimedean. The assumption that some $a \in]0,1[$ is an idempotent element of T implies $a_T^{(n)} = a$ for all $n \in \mathbf{N}$, so T can have only trivial idempotent elements.

If $\lim_{x \searrow x_0} T(x,x) = x_0$ for some $x_0 \in]0,1[$ and for all $y \in]x_0, l[$ we have $T(y,y) > x_0$ then, by induction, we also have

$y_T^{(n)} > x_0$

for all $y \in]x_0, l[$ and for all $n \in \mathbf{N}$, again violating (AP).

If T satisfies (iii), fix an arbitrary $x \in]0,1[$ and put $x_0 = \lim_{n\to\infty} x_T^{(n)}$. Then, because of the monotonicity (T3), we also have

$\lim_{y \searrow x_0} T(y,y) = x_0$.

If $x_0 > 0$ then there is some $y_0 \in]x_0, l[$ such that $T(y_0, y_0) = x_0$ and also $x_T^{(n)} < y_0$ for Some $n \in \mathbf{N}$, implying that we must have

$x_T^{(2n)} = x_0$

for all sufficiently large $n \in \mathbf{N}$, leading to the contradiction

$T(x_0, x_0) = x_T^{(4n)} = x_0$

Consequently, the only possibility is $x_0 = 0$. Since $x \in]0,1[$ was chosen arbitrarily, T satisfies (LP).

Finally, assume that T satisfies (ii) and choose $x, y \in]0,1[$. Because of $\lim_{n\to\infty} x_T^{(n)} = 0$, there exists an $n \in \mathbf{N}$ such that $x_T^{(n)} < y$

T is Archimedean, showing that (ii) implies (i).

The logical relationship between various algebraic properties of t-norms, a double arrow indicates an implication, a dotted arrow means that the corresponding implication holds for continuous t-norms.

In addition, Archimedean t-norms also have important properties, we only give the Theorem 10, and the other proofs will be omitted:

Theorem 9. Each left-continuous cancellative Archimedean t-norm is continuous.

In fact the calculative is not necessary. Here it can be weakened as follows:

Theorem 10. Each left-continuous Archimedean t-norm is continuous.

Proof:asssume that T is left-continuous and Archimedean , but not right-continuous in some point $(x_0, y_0) \in]0,1[^2$ .Then fix an arbitrary strictly increasing sequence $(z_n)_{n\in\mathbf{N}}$ in $[0,1]$ with $\lim_{n\to\infty} z_n = 1$.

Since T is Archimedean ,for each $n \in \mathbf{N}$ there exist numbers $k_n, l_n \in \mathbf{N}$ such that

$(z_n)_T^{k_n} \le x_0 < (z_n)_T^{k_n - 1}$,

$(z_n)_T^{l_n} \le x_0 < (z_n)_T^{l_n - 1}$.

implying that for all $n \in \mathbf{N}$

$(z_n)_T^{k_n + l_n} \le T(x_0, y_0) < T(x_0^+, y_0^+) \le (z_n)_T^{k_n + l_n - 2}$.

The left-continuity of T yields $\lim_{n\to\infty} (z_n)_T^{(2)} = 1$ and, consequently,

$\lim_{n\to\infty} T(T(x_0^+, y_0^+), (z_n)_T^{(2)}) = T(x_0^+, y_0^+)$

But then there is some $n \in \mathbf{N}$ such that

$(z_n)_T^{k_n + l_n} \le T(x_0, y_0) \ < T(T(x_0^+, y_0^+), (z_n)_T^{(2)}) \ \le T((z_n)_T^{k_n + l_n - 2}, (z_n)_T^{(2)}) \ = (z_n)_T^{k_n + l_n}$.

which is contradiction. Therefore, we have:

Theorem 11. Each cancellative Archimedean t-norm which is (left-)continuous in the point (1,1) is continuous.

Theorem 12. Let T be a continuous Archimedean t-norm. Then the following are equivalent:

(i) T is nilpotent.

(ii) There exists some nilpotent element of T.

(iii) There exists some zero divisor of T.

(iv) T is not strict.

The second characterizations to the Archimedean of t-norms

Theorem 1. T is right-continuous and has only trivial idempotent elements then it is Archimedean.

Proof:If a right-continuous t-norm T has only trivial idempotent elements then T satisfies (LP) by Proposition 2.6 (see [1]) and ,because of Theorem 2.12(see [1]),T is Archimedean.

It is easy to see T is continuous and has only trivial idempotent elements then it is Archimedean.

Theorem 2. If T is right-continuous and satisfies the CCL then it is Archimedean.

Proof: for an arbitrary $x \in]0,1[$ the sequence $(x_T^{(n)})_{n\in\mathbf{N}}$ converges to some $a \in [0,1]$ which is an idempotent element of T because of Proposition 2.6 (see [1]).Conversely, we have $T(a,a) = T(a,1)$ and, because of (CCL),a=0,i.e.,T has the LP and, because of Theorem 2.12 (see [1]), is Archimedean. Similarly,

Theorem 3. T is continuous and satisfies the CCL then it is Archimedean.

Theorem 4. T is continuous and satisfies the CL then it is Archimedean.

Proof: If t-norm T satisfies CCL,

then we can obtain y=z by

$T(x,y) = T(x,z) > 0$,

if $T(x,y) = T(x,z) = 0$, then we can obtain y=z or x=0.Since T has no zero divisor ,x can't be 0.Therefore T satisfies CL.

Definition 7. let T be a conditional calculative left-continuous t-norm which has no zero divisor, then T is not necessarily continuous. This example can show a conditional cancellative left-continuous t-norm Which is continuous in (1,1) is not necessarily continuous.

Ramark 1. If a t-norm T satisfies the CL then it obviously fulfills the CCL, but not conversely. In this sense ,we also obtain Theorem 4 .

Although Theorem 2 and Theorem.3 are correct for t-norms T, left-continuous conditional cancellative t-norm T is not necessarily Archimedean. For example, each $(x,y) \in]0,1]^2$ is in a one-to-one correspondence with a pair $((x_n)_{n\in N}, (y_n)_{n\in N})$ of strictly increasing sequences of natural numbers given by the unique infinite dyadic representations.

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{x_n}}$$

$$y = \sum_{n=1}^{\infty} \frac{1}{2^{y_n}}$$

of the numbers x and y, respectively .Using this notion ,then the t-norm $T:[0,1]^2 \to [0,1]$ given by

$$T(x,y) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{2^{x_n + y_n - n}} & (x,y) \in [0,1]^2 \\ 0 & \text{otherwise} \end{cases} \quad (*)$$

is strictly monotone ,therefor is calculative ,furthermore it is conditional cancellative, left-continuous on $[0,1]^2$ but it is neither Archimedean nor continuous

Theorem 5. If continuous t-norm T satisfies the CCL and has no zero divisor, then it is Archimedean.

All the implication between the algebraic properties of t-norms considered so far are summarized and visualized in Figure 1. Now we characterize some new properties.

Ramark 2. If t-norm T satisfies CCL and has no zero divisor, then T necessarily satisfies CL.

Proof: If t-norm T satisfies CCL, then we can obtain y=z by $T(x,y) = T(x,z) > 0$,if $T(x,y) = T(x,z) = 0$, then we can obtain y=z or x=0.

Since T has no zero divisor, x can't be 0.Therefore T satisfies CL. Thus let T be a conditional cancellative left-continuous t-norm which has no zero divisor, then T is not necessarily

continuous.(see(*)). This example can show a conditional calculative left-continuous t-norm Which is continuous in (1,1) is not necessarily continuous(This open
problem see[2]).

Theorem 6. T is continuous Archimedean t-norm and has no zero-divisors then T is strict.

Proof: Because of monotonicity , we have

$T(x,y) \le T(x,z)$

whenever $y < z$. T has no zero divisors, so

$T(x,y) \ne 0, T(x,z) \ne 0$

we will show $T(x,y) < T(x,z)$.

if $T(x,y) = T(x,z) > 0$,then T satisfies the CCL. as well known ,continuous Archimedean t-norm is either nilpotent or strict. If t-norm is strict ,obviously t satisfies the CL, then it satisfies the CCL.

If t-norm is nilpotent, the continuity of T implies that there exists a number $a \in [v, l[$ such that $y = T(z,a)$ Then

$$T(x,z) = T(x,y) = T(x, T(z,a))$$
$$= T(T(x,z),a) = \cdots = T(T(x,z), a_T^{(n)}),$$

for any $n \in \mathbf{N}$ .because of the nilpotent of T, we obtain $T(x,z) = 0$ .so T satisfies the CCL. so $y = z$,it violates the $y < z$. then $T(x,y) < T(x,z)$. e.g. T is strictly. Therefore T is strict.

Conclusions

we consider the Archimedean properties of t-norms. We summarize some Archimedean properties and characterize some new Archimedean properties of triangular norms on $([0,1], \le)$.

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