A Modified Natural BEM for Exterior Problems of the Helmholtz Equation

Quan Zheng[1,a], Yue Gao[1,b]

[1]College of Sciences, North China University of Technology, Beijing 100144, China

[a]zhengq@ncut.edu.cn,[b]846888286@qq.com

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Abstract. In this paper, a modified natural boundary element method (MNBEM) by using a modified DtN operator to improve the natural boundary element method (NBEM) is proposed for Neumann BVPs of the Helmholtz equation on unbounded domains. We prove the existence and the uniqueness of the solution of its variational problem in $L^2(\Gamma)$, and obtain L^2 error estimate of the boundary element solution which depends on the wave number k, the radius R of the boundary, the exact solution u, the mesh size h and the number N of truncated terms taken from the infinite series. The numerical results confirm the well-posedness and the convergence.

Introduction

Natural boundary integral equations for exterior Neumann BVPs can be solved directly by using the finite element method. That is the natural boundary element method (NBEM). K. Feng and D. Yu established the natural boundary integral equations by Green's functions on typical unbounded domains, then obtained boundary element solutions, and also applied the natural boundary integral equations in the coupling method and the domain decomposition method for the exterior problems [1-2]. R. Li obtained stiffness matrix of the NBEM corresponded to the linear element and the quadratic element for the exterior Helmholtz problem, and deduced error estimates [3]. Although sequences of absorbing boundary conditions with increasing accuracy and modified DtN (MDtN) boundary conditions were introduced into artificial boundary condition methods [4-6], and the error estimates depending on N, R and h were derived for the DtN-FEM coupling [7-9], the MDtN operator should be introduced into the NBEM for the exterior Helmholtz problem.

In this paper, a MNBEM for exterior Neumann BVPs of the Helmholtz equation is studied. We introduce the DtN operator and the MDtN operator in section 2; propose the MNBEM using a modified DtN operator, prove the uniqueness of solution of the variational problem and deduce the error estimate in section 3; present numerical examples in section 4; and finally make conclusions.

From a DtN operator to an MDtN operator

We consider the exterior Helmholtz problem

$$\Delta u + k^2 u = 0, \qquad \text{in} \ \ \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} = g, \qquad \text{on} \ \ \Gamma, \tag{2}$$

$$\frac{\partial u}{\partial r} - iku = o(r^{-\frac{1}{2}}), \qquad r \to \infty, \tag{3}$$

where Γ is a circle of radius R whose center is the origin of coordinates, Ω is an unbounded domain in the plane R^2 with boundary Γ, n is the outer normal to Γ and wave number $k > 0$.

The natural boundary integral equation of (1) (2) is

$$\frac{\partial u}{\partial n} = Ku(\theta), \qquad \text{on} \ \ \Gamma, \tag{4}$$

which is called a DtN operator, where $u(\theta) = u(R,\theta)$ and K is the boundary integral operator:

$$Ku(\theta) = \frac{k}{2\pi}\int_0^{2\pi}[\sum_{m=-\infty}^{+\infty} H_{|m|}(kR)\cos m(\theta - \theta')u(\theta')d\theta'], \tag{5}$$

$$H_m(z) = -\frac{\dfrac{dH_m^{(1)}(z)}{dz}}{H_m^{(1)}(z)} = \frac{H_{m+1}^{(1)}(z)}{H_m^{(1)}(z)} - \frac{m}{z}, \qquad m = 0,1,\cdots, \tag{6}$$

where $H_m^{(1)}(z)$ is the Hankel function of the first kind of order m.

Let $g \in L^2(\Gamma)$, then the variational problem corresponding to (4) is: Find $u \in H^1(\Gamma)$ such that

$$a(u,v) = <g,v>, \tag{7}$$

where $a(u,v) = <Ku,v>, <g,v> = \int_\Gamma g\,\bar{v}\,ds$.

Lemma 1. For all $z > 0$ and for all $m, m' \in R$ satisfying $|m| > |m'|$, we have

$$\operatorname{Im}(-H_{|m'|}(z)) > \operatorname{Im}(-H_{|m|}(z)) = \frac{2}{\pi z \,|H_{|m|}^{(1)}(z)|^2} > 0 \quad \text{and} \quad \operatorname{Re}(H_{|m|}(z)) > 0.$$

Lemma 2 The linear operator K: $H^s(\Gamma) \to H^{s-1}(\Gamma)$ defined by (4) satisfies

$$\sqrt{\tfrac{2\pi}{R}c_1}\|w\|_{s,\Gamma} \leq \|Kw\|_{s-1,\Gamma_0} \leq \sqrt{\tfrac{2\pi}{R}c_2}\|w\|_{s,\Gamma}, \tag{8}$$

where s is any real number, $c_1 = \inf\limits_{m\in N} |\dfrac{kRH_{|m|}(kR)}{\sqrt{1+m^2}}|^2 > 0$ and $c_2 = \sup\limits_{m\in N} |\dfrac{kRH_{|m|}(kR)}{\sqrt{1+m^2}}|^2$.

Theorem1. Let $g \in L^2(\Gamma)$, then the variational problem (7) has a unique solution $u \in H^1(\Gamma)$.

The problem (7) can be truncated into the following variational problem: Find $u^N \in H^1(\Gamma)$ s.t.

$$a^N(u^N,v) = <g,v>. \tag{9}$$

In fact, the truncation imposes an incorrect boundary condition $\frac{\partial u}{\partial n} = 0$ for the modes with $|m| > N$. To avoid the problems caused by truncation, Grote and Keller proposed the modified DtN boundary condition (see [5]).

According to the ideas, in this paper, we take

$$\frac{\partial u}{\partial n} = Bu \equiv -(ik - \tfrac{1}{2R})u, \quad \text{on } \Gamma, \tag{10}$$

the MDtN operator is defined by:

$$\frac{\partial u}{\partial n} = L^N u \equiv K^N u - \varsigma \sum_{|m|\leq N} u_m e^{im\theta} + \varsigma u, \quad \text{on } \Gamma \tag{11}$$

where $\varsigma = -ik + \frac{1}{2R}$, u_m is the Fourier coeffcient of $u(\theta)$, i.e. $u_m = \int_0^{2\pi} u(\theta)d\theta$.

We can see that B only replaces the action of K on the higher modes ($|m| > N$), without modifying the action of K on the lower modes. And by Lemma 1, the imaginary parts of $H_{|m|}(kR)$ and ζ are both negative, and the real parts of and $H_{|m|}(kR)$ and ζ are both positive.

A modified NBEM and its error estimate

Let $g \in L^2(\Gamma)$, then the variational problem corresponding to (7) is: Find $u^N \in H^1(\Gamma)$ such that

$$b^N(u^N,v) = <g,v> \tag{12}$$

where $b^N(u,v) = <L^N u,v>, \quad <g,v> = \int_\Gamma g\,\bar{v}\,ds$.

Lemma 3. The linear operator L^N: $H^s(\Gamma) \to H^{s-1}(\Gamma)$ satisfies

$$\|L^N w\|_{H^{s-1}(\Gamma)} \leq c_3 \|w\|_{H^s(\Gamma)} \quad \text{and} \quad \|L^N w\|_{L^2(\Gamma)} \leq c_3 \|w\|_{L^2(\Gamma)},$$

where $c_3 = \sqrt{2\pi R\max\{\max\limits_{|m|\leq N}|\dfrac{kH_{|m|}(kR)}{\sqrt{1+m^2}}|^2, \dfrac{|\varsigma|^2}{1+(1+N)^2}\}}$, $c_4 = \sqrt{2\pi R\min\{\min\limits_{|m|\leq N}|kH_{|m|}(kR)|^2,|\varsigma|^2\}}$ and s is any real number.

Theorem 2. Let $g \in L^2(\Gamma)$, then the variational problem (12) has a unique solution $u^N \in L^2(\Gamma)$.

Lemma 4. $\operatorname{Im}(-b^N(w,w))$ is $L^2(\Gamma)$-elliptic, i.e.

$$\text{Im}(-b^N(w,w)) \geq c_0' \parallel w \parallel_{L^2(\Gamma)}^2 \quad \text{and} \quad |b^N(w,w)| \geq c_0 \parallel w \parallel_{L^2(\Gamma)}^2,$$

where $c_0' = \min\{\frac{2}{\pi R |H_N^{(1)}(kR)|^2}, k\} > 0$, $c_0 = \frac{1}{\sqrt{2}} \min\{\min\limits_{|m| \leq N} |kH_{|m|}(kR)|, k + \frac{1}{2R}\} > 0$.

The variational problem corresponding to (7) is: Find $u_h^N \in H^1(\Gamma)$ such that

$$b^N(u_h^N, v_h) = \langle g, v_h \rangle, \qquad \forall v_h \in V_h(\Gamma). \tag{13}$$

Theorem 3. Let $g \in L^2(\Gamma)$, then the variational problem (13) has a unique solution $u_h^N \in L^2(\Gamma)$.

Let $g \in L^2(\Gamma)$, from Theorem 1, we known that (7) has a unique solution $u \in H^1(\Gamma)$. The numerical solution $u_h^N \in V_h(\Gamma) \subset H^1(\Gamma)$ by (13).

Theorem 4. Let $g \in L^2(\Gamma)$, u is the solution of (7). u_h^N is the solution of (13). Supposed that $u \in H^{p+1}(\Gamma)$, $p \geq 1$, and the finite element interpolation $\Pi : H^{p+1}(\Gamma) \to V_h$ satisfies $\parallel u - \Pi u \parallel_{H^1(\Gamma)} \leq C_0 h^p |u|_{p+1}$, then

$$\parallel u - u_h^N \parallel_{L^2(\Gamma)} \leq C_1 h^p |u|_{p+1} + C_2 R^N(u,k,R), \tag{14}$$

where $C_1 = C_0 + \dfrac{2C_0 \sqrt{\max\{\frac{\pi}{R} \max\limits_{|m| \leq N} |\frac{kRH_{|m|}(kR)}{\sqrt{1+m^2}}|^2, \frac{\pi(k^2+\frac{1}{4R^2})}{1+(1+N)^2}\}}}{k \min\{\min\limits_{|m| \leq N} |H_{|m|}(kR)|, 1+\frac{1}{2kR}\}}$, $C_2 = C_0 + \dfrac{\sqrt{2} \max\limits_{|m| > N} \frac{|kH_{|m|}(kR) - \varsigma|}{\sqrt{1+m^2}}}{k \min\{\min\limits_{|m| \leq N} |H_{|m|}(kR)|, 1+\frac{1}{2kR}\}}$,

$R^N(u,k,R) = (\sum\limits_{|m|>N}(1+|m|^2)|b_m|^2)^{\frac{1}{2}}$ and $b_m = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)e^{-im\theta} d\theta$.

Proof. $\forall v \in V_h$, from the variational problems (7), (12) and (13), we have

$$b^N(u_h^N - v, u_h^N - v) = b^N(u-v, u_h^N-v) - b^N(u, u_h^N-v) + a(u, u_h^N-v)$$

Because $u_h^N - v \in V_h$, let $u = \sum\limits_{m=-\infty}^{+\infty} b_m e^{im\theta}$, $u-v = \sum\limits_{m=-\infty}^{+\infty} c_m e^{im\theta}$ and $u_h^N - v = \sum\limits_{m=-\infty}^{+\infty} d_m e^{im\theta}$,

according to Lemma 4 and 3, we can obtain

$$\parallel u_h^N - v \parallel_{L^2(\Gamma)}^2 \leq c_0^{-1} |b^N(u_h^N - v, u_h^N - v)|$$
$$= c_0^{-1} |b^N(u-v, u_h^N-v) - b^N(u, u_h^N-v) + a(u, u_h^N-v)|$$
$$= c_0^{-1}(|\langle L^N(u-v), u_h^N-v \rangle| + |\langle (K-L^N)u, u_h^N-v \rangle|)$$
$$\leq \frac{c_3}{c_0} \parallel u-v \parallel_{H^1(\Gamma)} \parallel u_h^N-v \parallel_{L^2(\Gamma)} + \frac{1}{c_0} \max\limits_{|m|>N} \frac{|kH_{|m|}(kR)-\varsigma|}{\sqrt{1+m^2}} R^N(u,k,R) \parallel u_h^N-v \parallel_{L^2(\Gamma)}.$$

From the triangle inequality, we have

$$\parallel u - u_h^N \parallel_{L^2(\Gamma)} \leq \parallel u-v \parallel_{L^2(\Gamma)} + \parallel u_h^N - v \parallel_{L^2(\Gamma)} \leq (1+c_3 c_0^{-1}) \parallel u-v \parallel_{H^1(\Gamma)} + C_2 R^N(u,k,R).$$

So, we obtain

$$\parallel u - u_h^N \parallel_{L^2(\Gamma)} \leq (1+c_3 c_0^{-1}) \parallel u - \Pi u \parallel_{H^1(\Gamma)} + C_2 R^N(u,k,R) \leq C_0(1+c_3 c_0^{-1})h^p |u|_{p+1} + C_2 R^N(u,k,R).$$

The numerical example

Numerical results of examples are obtained by MNBEM as follows, where

$$\parallel u - u_h^N \parallel_{L^2(\Gamma)} = \sqrt{\sum_{i=1}^M (\theta_{i+1} - \theta_i)(u(x_i, y_i) - u_h^N(x_i, y_i))^2}, \quad Rate = \frac{\parallel u - u_h^N \parallel_{L^2(\Gamma)}}{\parallel u - u_{\frac{h}{2}}^N \parallel_{L^2(\Gamma)}}, \quad Order = \log_2 Rate.$$

Example. Let $\Omega = \{(x,y) \mid x^2 + y^2 > 225\}$ and $\Gamma = \partial\Omega$, consider the following equations

$$\begin{cases} \Delta u + k^2 u = 0, \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = e^{3i\theta} + e^{6i\theta} + e^{10i\theta}, \text{ on } \Gamma, \end{cases}$$

with the Sommerfeld radiation condition. This problem has an exact solution:

$$u(r,\theta) = -\frac{H_3^{(1)}(kr)}{kH_3^{(1)'}(15k)}e^{3i\theta} - \frac{H_6^{(1)}(kr)}{kH_6^{(1)'}(15k)}e^{6i\theta} - \frac{H_{10}^{(1)}(kr)}{kH_{10}^{(1)'}(15k)}e^{10i\theta}, r \geq 15, 0 \leq \theta < 2\pi.$$

Table 1. When $k = 1$, the errors of the MNBEM

Error	M	$N=5$	Rate	$N=6$	Rate	$N=10$	Rate	Order
$\| u - u_h^N \|_{L^2(\Gamma)}$	16	0.8669		1.5745		5.7568		
	32	0.3628	2.3896	0.3736	4.2145	0.4621	12.4590	3.6391
	64	0.0317	11.4351	0.0478	7.8031	0.1162	3.9774	1.9918
	128	0.0531	0.5978	0.0513	0.9313	0.0285	4.0724	2.0259
	256	0.0732	0.7250	0.0694	0.7385	0.0071	4.0208	2.0075

We find in Table 1 that when $N=10$, the MNBEM converges quadratically as $M \rightarrow \infty$, which verifies that C_1 is a constant and $R^N(u,k,R)=0$ in the error estimate for $N \geq 10$. When $N=5$ and 6, if M is not large enough, due to the accuracy of finite element is improved by increasing M the error of numerical solutions will decrease with the order of $O(M^{-2})$. If M is large, then the first part of the error in Theorem 4 will be very small and the second part will play a main role.

Table 2. When $k = 2$, the errors of the MNBEM

N	$M=120$		$M=200$	
	$\| u - u_h^N \|_{L^\infty(\Gamma)}$	$\| u - u_h^N \|_{L^2(\Gamma)}$	$\| u - u_h^N \|_{L^\infty(\Gamma)}$	$\| u - u_h^N \|_{L^2(\Gamma)}$
4	0.025830	0.049366	0.035184	0.068933
5	0.025830	0.049366	0.035184	0.068933
6	0.023777	0.048106	0.027899	0.065468
10	0.017457	0.032487	0.006258	0.011637
20	0.017457	0.032487	0.006258	0.011637
40	0.017457	0.032487	0.006258	0.011637

We see from Table 2 that the MNBEM is well-posed for any N.

Conclusions

We discuss the MNBEM for exterior Neumann boundary value problems of the Helmholtz equation in R^2 in this paper. The MNBEM is well-posed for any N. Moreover, we show that:

(1) when $N \geq m_{max}$, the MNBEM is second-order convergent with respect to h or $1/M$;

(2) when $N < m_{max}$, the MNBEM is nearly second-order convergent with respect to h when M is not large enough and almost holds on the accuracy when M is larger;

where m_{max} is the highest frequency of the exact solution.

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