Chaotic Behavior of Real Exchange Rate Model: Perspectives on Lyapunov Exponents

Chung-Ming Ou [1] **C.R. Ou [2]** **Nan-Chieh Chiu [3]**

[1]Department of Information Management, Kainan University, Taiwan
[2] Department of Electrical Engineering, Shu-Ping Institute of Technology, Taiwan
[3] Fixed Income Division, Polaris Securities, Taiwan

Abstract

The behaviors of the real exchange rate corresponding to capital mobility within an open-economy macroeconomic framework are investigated. Chaotic phenomena are reasonable candidates for analyzing the random walk behaviors of the given real exchange rate model similar to the simple logistic map. Method of Lyapunov exponents helps determine varied chaotic phenomena in order to choose specific capital mobility.

Keywords: real exchange rate, simple logistic function, chaos, bifurcation, Lyapunov exponent

1. Introduction

Studies of fluctuations of real exchange rate have attracted lots of efforts these days. According to Chen [1], these researches show that fluctuations in real exchange rates are results from fluctuations in underlying fundamental economic variables. The dynamic behavior of the exchange rate basically depends on two parameters, one is the trade balance elasticity, and the other one is the sensitivity of capital mobility. Chen [1] had deduced a cubic equation to simulate this real exchange rate model. This simple trade model modified by incorporating capital mobility generates chaotic phenomena of the real exchange rate, which is also totally independent of underlying economic fundamentals.

A country's real exchange rate is defined as the relative purchasing power of domestic output, namely, the term of trade in trade theory. Instantaneous adjustment in asset markets and the perfect mobility of capital ensures that the real exchange rate stays at its equilibrium state. However, if capital mobility is not perfect, periodic and chaotic situations can be caused by capital market imperfection even when prices are perfectly flexible. In this case, the dynamics of exchange rate will not converge asymptotically to the equilibrium state for almost all initial conditions; instead, the dynamics may converge to a periodic orbit or even to a chaotic attractor. That is, regular period doubling bifurcations occur first, and then come the chaotic behaviors.

The phenomenon of chaos gives a nature way to describe the above observations of real exchange rates. A function f is said to be *chaotic*, if it is sensitively dependence on initial conditions (SIC). This means small perturbations in the initial values will influence the function behaviors greatly; SIC leads to the unpredictability via initial conditions, which is the basis for randomness [2]. This method is also proposed in [1]. One of the most famous chaotic functions is so-called the simple logistic function (SLF), which is a one-parameter family of quadratic functions. For chaotic functions, one important issue is to find a pair of proper initial value x_0 and function parameter r such that the (r, x_0) falls into the so-called ***chaotic region*** which is a subset of Euclidean space $\mathbf{R}^2$. We investigate this issue via the method of Lyapunov exponents, which can characterize the degree of chaos and lead to the methodology of testing chaos. Chen's model of the real exchange rate model is a cubic equation showing the similar behavior as SLFs.

The arrangement of this paper is as follows. In section 2, we describe some concepts of chaos such as chaotic functions, SLFs and Lyapunov exponents. In section 3, we investigate chaotic behaviors of the real exchange rate model via the method of Lyapunov exponents.

2. Background

2.1. Chaotic Functions

A function f is ***chaotic*** if the following conditions are met:

(1) Sensitivity to initial Conditions (SIC): Two starting states very close to each other will yield series of iterations that will differ significantly after a while.
(2) f is nonlinear and difficult or impossible to predict analytically.
(3) The dynamics of f show more and more complex detail as examination of it scales down. A plot of the system dynamics thus shows finer structure the more we room in.

One of the standard chaotic functions is related to the population dynamics, called simple logistic function (**SLF**). This is a simple one-parameter second order function, which shows a prosperous chaotic behavior for different parameter; it is defined as

$$f(x)=rx(1-x) \quad (1)$$

where $0 \leqq x \leqq 1;\ 0 \leqq r \leqq 4$. We call r the bifurcation parameter of SLF. f defines a dynamical system via the ***iteration,*** which is simply

$$x_{n+1} = rx_n(1-x_n) \quad (2)$$

for n=0,1,2,… . We call it the ***SLF-based iteration***. We illustrate the function behavior in the Figure 2.1. We observe that while r = 3, the iteration sequence splits into a two-periodic oscillation, which continues for r up to about 3.45; the split is called a ***bifurcation*** from chaos sciences. Successive doublings of the period then occurs fast in the approximation range 3.55 < r < 3.6. After r = 3.6, the periodicity have transferred to chaos.

From the figure 1 and 2, the dark area to the right of approximately r = 3.55 is called the chaotic region. SLF with parameter r is sensitive to the perturbation of the initial value xo, if (r, x_0) falls into the chaotic region. However, we see a bright strip around r = 3.83. Therefore, we don't conclude that any initial condition with r > 3.6 falls into the chaotic region; we need more tools, such as method of Lyapunov exponents, to justify this.

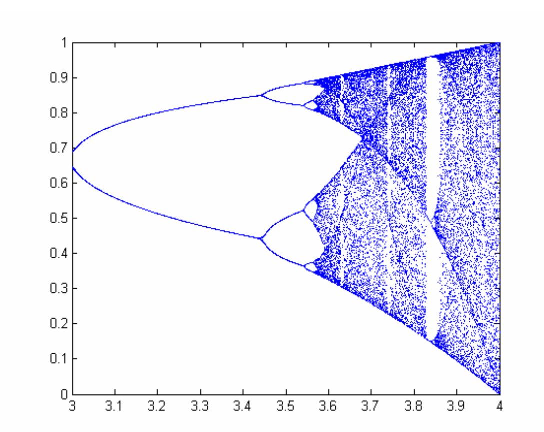

Figure 1: Chaotic Behavior of SLF for 3< r < 4.

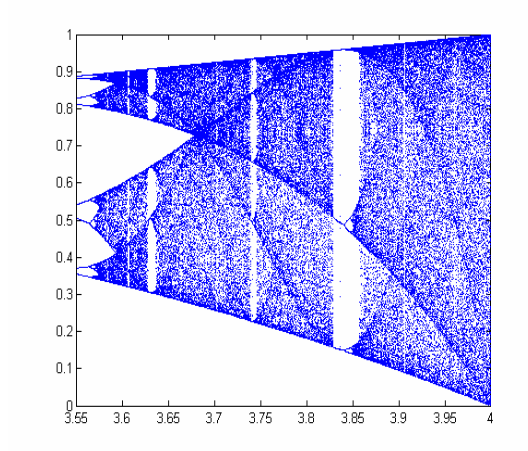

Figure 2: Chaotic Behavior of SLF for 3.55< r < 4.

2.2. Method of Lyapunov Exponents

The following question arises: how to choose a proper $(r, x_0) \in [0\ 4] \times [0\ 1]$ such that it falls into the chaotic region. A common technique to measure chaos is to compute the ***Lyapunov Exponents*** of the iteration (2). In general, given a differentiable function f(x), the Lyapunov exponent λ of f at x_0 is as follows:

$$\lambda(x_0) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right) \quad \textbf{(3)}$$

For SLFs (1), (3) can be reduced as

$$\lambda(x_0;r) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\ln|r-2rx_i|\right) \quad \textbf{(4)}$$

Lyapunov exponents represent the strength of the sensitivity to the initial conditions. For chaotic dynamical systems, Lyapunov exponents are independent of the initial value x_0, as long as x_0 is not periodic. A positive Lyapunov exponent indicates that the measured iteration is sensitive to initial conditions, see Figure 3.

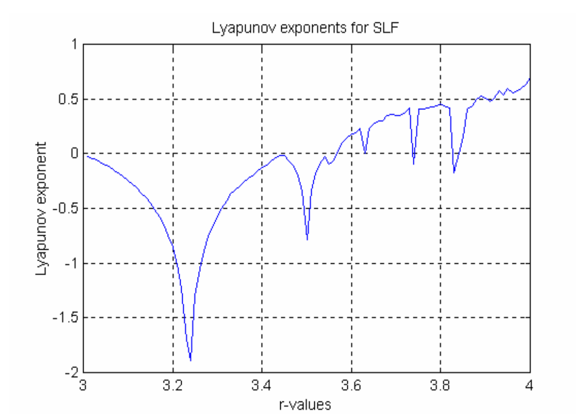

Figure 3: Lyapunov Exponents of SLFs corresponding to chaotic region.

3. Chaotic Behaviors of Real Exchange Rate

According to Chen [1], the real exchange rate, q_n , can be written as the following one-dimensional nonlinear cubic equation:

$$q_{n+1} = f(q_n, r) \quad (5)$$

$$f(q_n, r) = (bq_n^3 - aq_n^2 + cq_n - d)/r \quad (6)$$

Where r is the bifurcation parameter and n is the time iteration. From exchange rate model, r represents the speed of adjustment in asset markets caused by international gaps in real returns. When the capital mobility is perfect, $r \to \infty$. Now we simulate (5)-(6) for r varied between 0.35 and 0.55 horizontally. On the vertical axis were plotted all iterated real exchange rates q_n from n = 0 to n = 10000. The result is shown in Figure 4 under the fixed parameters suggested by Chen [1] with a = 7, b = 2, c = 7, d = 2 and an initial exchange rate of 0.85. It shows the usual period doubling bifurcation pattern and some short-period windows of as SLFs in Figure 1.

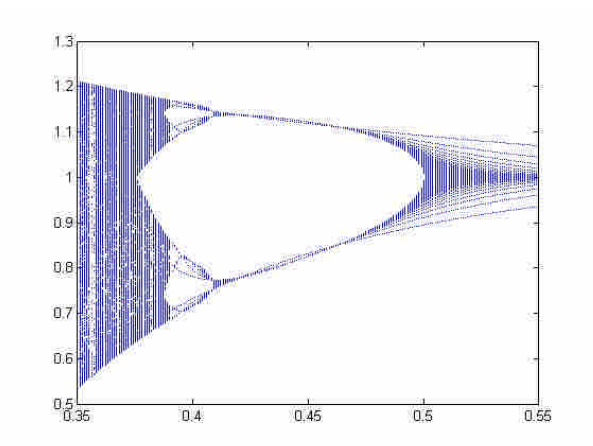

Figure 4: Bifurcation diagram of Real exchange rate model

We test the chaotic behavior of exchange rate model (5)-(6) via the method of Lyapunov exponents described in subsection 2.2. We realize the chaotic behaviors do happen as r runs from 0.39 to 0.35, where Lyapunov exponents are positive, see Figure 5. On the other hand, when r >0.4, the real exchange rates show periodic doubling phenomena without chaos until r is close to 0.5, where Lyapunov exponent is around zero.

From the above simulation, we learn that when the capital mobility becomes smaller, more cycles appear. However, if the speed of adjustment in asset markets is low, that is, if the cost of intertemporal adjustment through international borrowing and lending is high, there is no way for the economy to arrive at the deterministic intertemporal equilibrium (for example, r<0.355)

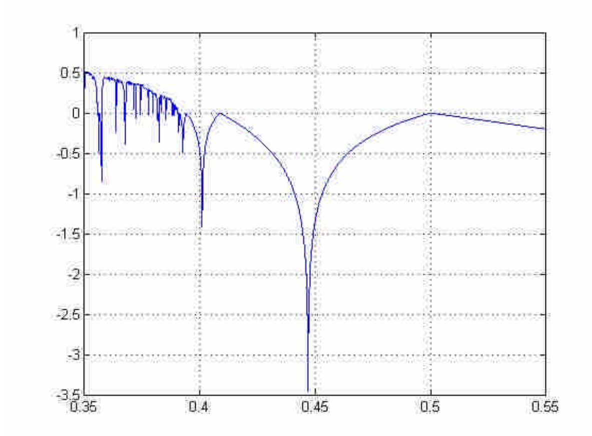

Figure 5: Lyapunov exponents of real exchange rate model (5)-(6).

However, we need to be careful that Lyapunov exponents are also fluctuated above and below zero when 0.35<r<0.4; it means that choosing proper capital mobility is more than an initiative task, while necessary methodologies such as method of Lyapunov exponents is important.

4. Conclusions

We propose the real exchange rate model similar to simple logistic functions, which shows prosperous chaotic behaviors while the parameter varied. This bifurcation parameter is related to capital mobility. From numerical simulation point of view, the Lyapunov exponents can detect chaotic behaviors effectively and help determine proper capital mobility for stable equilibrium exchange rate.

5. References

[1] Shikuan Chen, “Complex Dynamics of the Real Exchange Rate in an Open Macroeconomic Model”, *Journal of Macroeconomics*, summer 1999, Vol.21, No.3, pp.493-508.

[2] M. Svensson and J-E. Malmquist, “A Simple Secure Communications System Utilizing Chaotic Functions to Control the Encryption and Decryption of Messages”, Project report for the course ‘Chaos in science and technology’, Lund Institute of Technology, Dept. of Physics, sub-dept. of mathematical physics, 1996.