Constructing Fusion Frames with General Shifts

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Abstract—Frame theory plays an important role in the signal processing and image processing. In this paper, symmetric fusion frames with general shift are constructed, which generalizes the existing results of integer shift to the general case.

Keywords-frame;wavelet; fusion frame; symmetry property; general shift

I. INTRODUCTION

Frame theory was first introduced and discussed by Duffin and Schaeffer [1] when they studied nonharmonic Fourier series in 1952. Since the excellent work of Daubechies, Grossmann, and Meyer [2] is published, the theory of frame began to be widely studied all over the world and was used to all kinds of the engineering and technology field [3-6].

An important example about frame is wavelet frame [7], which is obtained by shifting and dilating a mother wavelet function or a finite family of mother wavelet functions. Since wavelets were introduced in the 1980s, they attracted considerable interest from the mathematical community and many other diverse disciplines. Wavelets had promising applications in the signal processing and image processing and had a huge success in the engineering and technology field. Therefore, wavelet plays an important role in all mathematical, engineering and other fields in the last decade and few other mathematical theoretical sciences have enjoyed this much attention and popularity.

Daubechies firstly developed the wavelet frame theory at an expense of some redundancy property and gave the transition from continuous signal analysis to discrete signal analysis. She constructed all kinds of elegant and simple wavelet orthonormal functions by a novel way. They have become the corner stone of wavelet applications nowadays. In particular, in 1989, Mallat [8] developed the new important theory of multiresolution analysis (MRA) for discrete wavelet transform (DWT) and gave excellent fast algorithm. He obtained a huge improvement through his work in digital signal processing and image processing and first discovered some relationships among pyramid algorithms, quadrature mirror filters and orthonormal wavelet bases.

Wavelet theory has been studied extensively in both theory and applications since 1980's. The basic advantage of wavelet is that they can be simultaneously described signal in both the frequency domain and the time domain. Furthermore, wavelet has time-frequency localization property, make it more better deal with signal. Recently, many people found that wavelets do not perform to represent and analyze anisotropic features in multivariate signal processing. This promotes people to look for better mathematical tools.

Another most important concrete realization of frame is Gabor frame. Gabor systems were first introduced and discussed by Gabor [9]. They are generated by modulations and translations of a finite family of functions. At first, people devoted to studying some important characterizations and constructions about Gabor frames with regular sampling lattices. Then, other authors also discussed Gabor frames with irregular sampling lattices and the case of higher dimensions. In their papers, the irregularity sampling means to either the sampling lattice set or the adjusted windows. However, few people paid attention to irregular Gabor frames. Recently, they gave necessary and sufficient conditions about Gabor frame generators in some conditions and presented some open problems. Furthermore, some people provided several new results for semiirregular Gabor frames and gave new proof. The book of Grochenig [10] also gives an elaborate description about time frequency analysis and Gabor systems and provides many good examples.

It is well known that the symmetric wavelet plays an important role in signal processing and image processing. In [11], authors gave a simple but useful way to construct symmetric or antisymmetric wavelet frames from any given wavelet frames. Motivated by the above way, some authors discussed the case of corresponding Gabor frames [12]. Symmetric Gabor frames with single generator about origin and about integer sampling lattices are constructed. In particular, symmetric Parseval Gabor frames are provided. At last, some good examples are given to prove their theory. Recently, L. F. Wang and Y. Liu [13] gave a similar way to construct of symmetric Gabor frames with several generators in regular sampling lattices in higher dimensions.

In paper [14], authors gave the definition of wave packet systems by fusing three operators of dilations, modulations and translations to the Gaussian function when studying some classes of singular integral operators. In paper [15], authors used the same expression to describe any collections of functions composed of the same three operations. That is, let $g \in L^2(R)$ and define the associated wave packet system as the following set

$$G(g, j, l, k) = \{D_2^{\ j} E_{al} T_{bk} g : j, l, k \in Z\}, \qquad (1.1)$$

where $D_2 f(x) = \sqrt{2} f(2x),\ T_k f(x) = f(x-k)$ and $E_l f(x) = e^{2\pi l x} f(x)$, a>0,b>0. In fact, wave packet system is generalization of Gabor systems, wavelet systems and the Fourier transform of wavelet systems. Thus, in this paper, we name wave packet systems as fusion systems and call wave packet frames as fusion frames. Fusion systems have obtained huge success in harmonic analysis and operator theory [16, 17]. In paper [18], authors dealt with wave packet systems as special cases of generalized shift invariant systems and gave a sufficient condition of a wave packet frame. In paper [19], authors gave a unified dealt and provided more examples of wave packet frames.

In paper [20], authors discussed some characterizations of the multivariate wave packet systems. They provided the characterizations of the orthogonal wave packet systems and obtained the necessary conditions and sufficient conditions for the wave packet systems to be wave packet Parseval frames with the very general lattices are. Furthermore, they [21] established the necessary conditions and sufficient conditions for all kinds of wave packet frames of the different operator order in $L^2(R^n)$ with an arbitrary expanding matrix, which include the corresponding results of wavelet analysis and Gabor theory as the special cases. Recently, the paper [22] provides a characterization of wave packet Parseval frames from their Fourier transforms, which deduce some existing results as the corollaries. Furthermore, based on the characterization of wave packet Parseval frames, some properties and a sufficient condition of wave packet frame multipliers are obtained.

Except for above three systems that we have mentioned, wavelet systems with composite dilation and shearlet systems have widely studied and developed by people. People can refer to the review paper [23] for more knowledge about all kinds of reproducing systems, which presented an overview of all kind of reproducing systems which are obtained by applying a combination of dilations, modulations and translations to a finite family of functions consisting of Gabor systems, wavelet systems, wave packet systems, composite dilation wavelet systems and shearlet systems. Authors reviewed their definitions, history and existing known results, respectively. Furthermore, They also discuss their advantages and shortcomings in the engineering applications. The fields that are discussed above is called applied harmonic analysis, which plays an important role in engineering such as signal processing, image processing, digital communications, medical imaging, and so on.

Motivated by the existing way [11], G. C. Wu and H. X. Cao [24] constructed symmetric fusion systems. Their way combines with some techniques in wavelet analysis and time-frequency analysis. This way makes the amount of wavelets largely increase. In the paper [25], symmetric fusion frames with several generators are constructed from any fusion frames given, which generalizes the existing result in [24] to the case of several generators. In this paper, symmetric fusion frames with general shift are constructed, which generalizes the existing results of integer shift to the general case.

II. PRELIMINARIES

Throughout this paper, the following notations will be used. R and Z denote the set of all real numbers and the set of all integers, respectively. $L^2(R)$ is the space of all square-integrable functions, and $<\cdot,\cdot>$ and $\|\ \|$ denote the inner product and norm in $L^2(R)$, respectively, and $l^2(Z)$ denotes the space of all square summable sequences. Let us recall the definition of frame.

Definition 2.1 Let H be a separable Hilbert space. A sequence $\{f_i\}_{i\in N}$ of elements of H is a frame for H if there exist constants $0 < C \le D < \infty$ such that for all $f \in H$,we have

$$C\|f\|^2 \le \sum_{i\in N} \left\|\langle f, f_i\rangle\right\|^2 \le D\|f\|^2. \qquad (2.1)$$

The numbers C, D are called lower and upper frame bounds, respectively (the largest C and the smallest D for which (2.1) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (2.1) are called Bessel sequences.

Let T_f denote the synthesis operator of $f = \{f_i\}_{i\in N}$, i.e., $T_f(c) = \sum_i c_i f_i$ for each sequence of scalars $c = (c_i)_{i\in N}$. Then the frame operator $Sh = T_f T_f^*(h)$ associated with $\{f_i\}_{i\in N}$ is a bounded, invertible, and positive operator mapping of H on itself. This provides the reconstruction formula

$$h = \sum_{i=1}^{\infty} <h, g_i> f_i = \sum_{i=1}^{\infty} <h, f_i> g_i, \forall\ h \in H. \quad (2.2)$$

where $g_i = S^{-1} f_i$. The family $\{g_i\}_{i\in N}$ is also a frame for H and is called the canonical dual frame of $\{f_i\}_{i\in N}$. If $\{g_i\}_{i\in N}$ is any sequence in H which satisfies

$$h = \sum_{i=1}^{\infty} <h, g_i> f_i = \sum_{i=1}^{\infty} <h, f_i> g_i, \forall\ h \in H, \quad (2.3)$$

it is called an alternate dual frame of $\{f_i\}_{i\in N}$.

Then, we will give the definitions of a fusion frame and the fusion frame function.

Definition 2.2 We say that the fusion system defined by (1.1) is a fusion frame if it is a frame for $L^2(R)$. Then, the function g is called a fusion frame function.

III. MAIN RESULTS

In this following, we will construct symmetric wave packet frames from any given fusion frames by making use of existing way. For a function $g\in L^2(R)$, we can define new symmetric or antisymmetric functions about origin as the following:

$$g_1(x)=\frac{g(x)+g(-x)}{2},\quad g_2(x)=\frac{g(x)-g(-x)}{2}. \tag{3.1}$$

Thus, we have

Theorem 3.1 Suppose that fusion system

$$\{D_2{}^jE_{al}T_{bk}g: j\in Z,\ k\in Z,\ l\in Z\}$$

defined by (1.1) is a frame for $L^2(R)$ with frame bounds C_1,C_2, then fusion system

$$\{D_2{}^jE_{al}T_{bk}g_1 \cup D_2{}^jE_{al}T_{bk}g_2 : j\in Z, k\in Z,\ l\in Z\} \tag{3.2}$$

is a symmetric or antisymmetric frame for $L^2(R)$ about origin with frame bounds C_1,C_2, where the functions $g_1(x), g_2(x)$ are defined by (3.1).

Proof. Since fusion system

$$\{D_2{}^jE_{al}T_{bk}g: j\in Z, k\in Z, l\in Z\}$$

is a frame with bounds C_1,C_2, for all $f(x)\in L^2(R)$, we have

$$C_1\|f\|^2\le\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f, D_2{}^jE_{al}T_{bk}g\rangle|^2\le C_2\|f\|^2. \tag{3.3}$$

In order to prove that fusion system defined by (3.2) is a frame, we firstly calculate the series

$$\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f, D_2{}^jE_{al}T_{bk}g_1\rangle|^2 +\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f, D_2{}^jE_{al}T_{bk}g_2\rangle|^2. \tag{3.4}$$

According to definition of g_1 and the property of inner product, we can obtain

$$|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g_1\rangle|^2= |\langle f(\cdot), D_2{}^jE_{al}T_{bk}\frac{g(\cdot)}{2}\rangle+\langle f(\cdot), D_2{}^jE_{al}T_{bk}\frac{(-\cdot)}{2}\rangle|^2. \tag{3.5}$$

For any complex numbers z_1, z_2, it is well known that the following equality holds

$$|z_1+z_2|^2=|z_1|^2+|z_2|^2+z_1\overline{z_2}+\overline{z_1}z_2. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g_1(\cdot)\rangle|^2&=\frac{1}{4}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle|^2\\&+\frac{1}{4}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle|^2\\&+\frac{1}{4}\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle\overline{\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle}\\&+\frac{1}{4}\overline{\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle}\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle.\end{aligned} \tag{3.7}$$

In the similar way, we can obtain

$$\begin{aligned}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g_2(\cdot)\rangle|^2&=\frac{1}{4}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle|^2\\&+\frac{1}{4}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle|^2\\&-\frac{1}{4}\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle\overline{\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle}\\&-\frac{1}{4}\overline{\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle}\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle.\end{aligned} \tag{3.8}$$

Comparing with (3.7) and (3.8), we have

$$\begin{aligned}&\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f, D_2{}^jE_{al}T_{bk}g_1\rangle|^2 +\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f, D_2{}^jE_{al}T_{bk}g_2\rangle|^2\\&=\frac{1}{2}\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f(\cdot), D_A{}^jE_{al}T_{bk}g(\cdot)\rangle|^2\\&+\frac{1}{2}\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle|^2.\end{aligned} \tag{3.9}$$

By simple calculation, get

$$\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f(\cdot), D_2{}^jE_{al}T_{bk}g(-\cdot)\rangle|^2 =\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f(-\cdot), D_2{}^jE_{al}T_{bk}g(\cdot)\rangle|^2. \tag{3.10}$$

According to (3.3), we get

$$C_1\|f(-\cdot)\|^2\le\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z}|\langle f(-\cdot), D_2{}^jE_{al}T_{bk}g\rangle|^2 \le C_2\|f(-\cdot)\|^2, \tag{3.11}$$

From (3.10), (3.11) and the equality

$$\|f(-\cdot)\|^2=\|f(\cdot)\|^2,$$

we deduce

$$C_1\|f\|^2 \le \sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z} |\langle f, D_2{}^j E_{al} T_{bk} g(-\cdot)\rangle|^2$$
$$\le C_2\|f\|^2, \ \forall f(x)\in L^2(R). \tag{3.12}$$

At last, comparing with (3.3), (3.9) and (3.12), we obtain

$$C_1\|f\|^2 \le \sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z} |\langle f, D_2{}^j E_{al} T_{bk} g_1\rangle|^2$$
$$+\sum_{j\in Z}\sum_{l\in Z}\sum_{k\in Z} |\langle f, D_2{}^j E_{al} T_{bk} g_2\rangle|^2 \le C_2\|f\|^2. \tag{3.13}$$

Therefore, we have completed the proof of Theorem 3.1.

Remark

(1) In particular, let $a=b=1$ in the Theorem 3.1, then, we obtain the construction of symmetric fusion frames with integer shift from any wavelet frames given, which have been proved in [24].

Corollary 3.1. Suppose that fusion system

$$\{D_2{}^j E_l T_k g : j\in Z, k\in Z, l\in Z\}$$

is a frame for $L^2(R)$ with bounds C_1, C_2, then fusion system

$$\{D_2{}^j E_l T_k g_1 \cup D_2{}^j E_l T_k g_2 : j\in Z, k\in Z, l\in Z\}$$

is a symmetric frame for $L^2(R)$ about origin with frame bounds C_1, C_2, where the functions $g_1(x), g_2(x)$ are defined by (3.1).

IV. Summary

All kings of reproducing systems have obtained huge success in image compression, edge analysis, detection and other fields. In this paper, symmetric fusion frames with general shift are constructed from wave packet frames given, which generalizes the existing results of integer shift to the general case.

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