The Periodic Solution of an Impulsive Perturbed Two-Species Gilpin-Ayala Competition System

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Abstract—The principle purpose of this paper is to explore the existence of periodic solution of Two-Species Gilpin-Ayala competition system with impulsive perturbations. Sufficient and realistic conditions are obtained by using Mawhin's continuation theorem of the coincidence degree. Further, some numerical simulations show that our model can occur in many forms of complexities including periodic oscillation and Gui chaotic strange attractor.

Periodic solution; competitive system ; Gilpin-Ayala system; impulsese; coincidence degree theory;

I. INTRODUCTION

The dynamics of Ayala-Gilpin competitive system, which was first introduced by Ayala et al. [1], has been widely studied by many authors [2-6]. However, the corresponding problems with periodic coefficients and impulsive perturbations were studied far less often [7]. In this paper, we will study the following impulsive Gilpin-Ayala system:

$$\begin{cases} \left.\begin{aligned} y_1'(t) &= r_1(t)y_1(t)\left[1-\left(\frac{y_1(t)}{K_1(t)}\right)^{\theta_1} - \alpha_{12}(t)\frac{y_2(t)}{K_2(t)}\right] \\ y_2'(t) &= r_2(t)y_2(t)\left[1-\left(\frac{y_2(t)}{K_2(t)}\right)^{\theta_2} - \alpha_{21}(t)\frac{y_1(t)}{K_1(t)}\right] \end{aligned}\right\} t \neq t_k \\ \left.\begin{aligned} \Delta y_1(t_k) &= y_1(t_k^+) - y_1(t_k^-) = p_{1k}y_1(t_k^-) \\ \Delta y_2(t_k) &= y_2(t_k^+) - y_2(t_k^-) = p_{2k}y_2(t_k^-) \end{aligned}\right\} t = t_k,\ k \in 1,2,\ldots \end{cases} \quad (1)$$

where $y_i(t)$ represents the density of the i th species at time t ; $r_i(t)$ denotes the intrinsic growth rate of the i th species; $K_i(t)$ means the environment carrying capacity of species i in the absence of competition; $\alpha_{ij}(t)$ ($i \neq j$) measures the amount of competition between the species y_i and y_j ; θ_i is a positive constant and provide a nonlinear measure of intra-specific interference; p_{ik} are constants, $i = 1,2$.

In system (1), we give two hypotheses as follows.

(H1) $r_1(t), r_2(t), K_1(t), K_2(t), \alpha_{12}(t)$ and $\alpha_{21}(t)$ are all nonnegative ω - periodic functions defined on $\mathbb{R}$.

(H2) $1 + p_{ik} > 0$ and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $p_{i(k+q)} = p_{ik}$, $i = 1,2$.

II. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

To prove our results, we need the notion of Mawhin's continuation theorem formulated in [8].

Lemma 1 ([8]). Let X and Y be two Banach spaces. Consider an operator equation $Lx = \lambda Nx$ where $L: \mathrm{Dom}\, L \cap X \to Y$ is a Fredholm operator of index zero and $\lambda \in [0,1]$ is a parameter, then there exist two projectors $P: X \to X$ and $Q: Y \to Y$ such that $\mathrm{Im}\, P = \ker L$ and $\mathrm{Im}\, L = \ker Q$. Assume that $N: \bar{\Omega} \to Y$ is L - compact on $\bar{\Omega}$, where Ω is open bounded in X . Furthermore, assume that

(a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \mathrm{Dom}\, L$, $Lx \neq \lambda Nx$;

(b) for each $x \in \partial\Omega \cap \ker L$, $QNx \neq 0$;

(c) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where

$J: \mathrm{Im}\, Q \to \ker L$ is an isomorphism and $\deg\{*\}$ represents the Brouwer degree.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{Dom}\, L$.

For the sake of convenience, we shall make some preparation. Let $I \subset R$. Denote by $PC(I, \mathbb{R}^2)$ the space of functions $x(t): I \to R^2$ which are continuous at $t \in I$, $t \neq t_k$, and are left continuous for $t = t_k \in I$. Let

$$u^L = \min_{0 \le t \le \omega}\{u(t)\},$$
$$u^M = \max_{0 \le t \le \omega}\{u(t)\},$$
$$\bar{u} = \frac{1}{\omega}\int_0^\omega u(t)dt,$$
$$\overline{uv} = \frac{1}{\omega}\int_0^\omega u(t)v(t)dt,$$

where $u(t)$, $v(t)$ are ω -periodic functions.

Theorem 1. Suppose (H1) and (H2) hold, furthermore, the following conditions are satisfied.

(H3) $\sum_{k=1}^{q} \ln(1 + p_{1k}) + \bar{r}_1\omega > \frac{1}{K_2^L}\overline{r_1\alpha_{12}}\omega e^{C_2}$, (H4) $\sum_{k=1}^{q}\ln(1+p_{2k})+\overline{r}_2\omega>\frac{1}{K_1^L}\overline{r_2\alpha_{21}}\omega e^{C_1}$,

where

$$C_i=\frac{1}{\theta_i}\ln[1+\frac{1}{\overline{r}_i\omega}\sum_{k=1}^{q}\ln(1+p_{ik})]+\ln K_i^M+2\overline{r}_i\omega$$
$$+\sum_{k=1}^{q}\ln(1+p_{ik})+\sum_{k=1}^{q}|\ln(1+p_{ik})|,\quad i=1,2.$$

Then system (1) has at least one positive ω-periodic solution.

Proof. Let

$$y_i(t)=e^{x_i(t)},\quad i=1,2 \tag{2}$$

then system (1) becomes

$$\begin{cases}\left.\begin{aligned}x_1'(t)&=r_1(t)\left[1-\frac{e^{\theta_1x_1(t)}}{K_1^{\theta_1}(t)}-\alpha_{12}(t)\frac{e^{x_2(t)}}{K_2(t)}\right]\\ x_2'(t)&=r_2(t)\left[1-\frac{e^{\theta_2x_2(t)}}{K_2^{\theta_2}(t)}-\alpha_{21}(t)\frac{e^{x_1(t)}}{K_1(t)}\right]\end{aligned}\right\}t\neq t_k\\ \left.\begin{aligned}\Delta x_1(t_k)&=\ln(1+p_{1k})\\ \Delta x_2(t_k)&=\ln(1+p_{2k})\end{aligned}\right\}t=t_k\end{cases} \tag{3}$$

In order to use Lemma 1, we set

$$x=(x_1(t),x_2(t))^T,$$
$$X=\{x\in PC(\mathbb{R},\mathbb{R}^2)\mid x(t+\omega)=x(t)\},$$
$$Y=X\times R^{2q},$$

then it is standard to show that both X and Y are Banach space when they are endowed with the norm

$$\|x\|_c=\sup_{t\in[0,\omega]}|x(t)|$$

and

$$\|(x,c_1,\cdots,c_q)\|=(\|x\|_c^2+|c_1|^2+\cdots+|c_q|^2)^{1/2}.$$

Set $L:\mathrm{Dom}\,L\subset X\to Y$ as

$$(Lx)(t)=(x'(t),(\Delta x(t_1),\cdots,\Delta x(t_q))),$$

where

$$\mathrm{Dom}\,L=\{x\in X\mid x'(t)\in PC(\mathbb{R},\mathbb{R}^2)\}$$
$$\mathrm{Im}\,L=\left\{(y,c_1,\cdots,c_q)\in Y\mid \int_0^{\omega}y(t)dt+\sum_{i=1}^{q}c_i=0\right\}$$

and

$$\ker L=\mathbb{R}^2.$$

At the same time, we denote $N:X\to Y$ as

$$(Nx)(t)=(f(t,x(t)),\Phi_1(x(t_1)),\cdots,\Phi_q(x(t_q))),$$

where

$$f(t,x)=\begin{bmatrix}r_1(t)\left[1-\frac{e^{\theta_1x_1(t)}}{K_1^{\theta_1}(t)}-\alpha_{12}(t)\frac{e^{x_2(t)}}{K_2(t)}\right]\\ r_2(t)\left[1-\frac{e^{\theta_2x_2(t)}}{K_2^{\theta_2}(t)}-\alpha_{21}(t)\frac{e^{x_1(t)}}{K_1(t)}\right]\end{bmatrix},$$
$$\Phi_k(x(t_k))=\begin{bmatrix}\ln(1+p_{1k})\\ \ln(1+p_{2k})\end{bmatrix},\ k=1,2,\cdots,q.$$

Define two projectors P and Q as

$$P:X\to\ker L,$$
$$Px=\frac{1}{\omega}\int_0^{\omega}x(t)dt,$$
$$Q:Y\to Y,$$
$$Q(y,c_1,\cdots,c_q)=\frac{1}{\omega}\left(\int_0^{\omega}y(s)ds+\sum_{k=1}^{q}c_k,0,\cdots,0\right).$$

It can be easily proved that L is a Fredholm operator of index zero; P, Q are projectors; and N is L-compact on $\overline{\Omega}$ for any given open and bound subset Ω in X.

Now we are in a position to search for an appropriate open bounded subset Ω for the application of Lemma 1 corresponding to operator equation

$$Lx=\lambda Nx,\quad \lambda\in(0,1). \tag{4}$$

Suppose that $x(t)=(x_1(t),x_2(t))^T$ is a periodic solution of (4) for certain $\lambda\in(0,1)$. By integrating (4) over the interval $[0,\omega]$, we get

$$\begin{aligned}\overline{r}_1\omega&=-\sum_{k=1}^{q}\ln(1+p_{1k})+\int_0^{\omega}\frac{r_1(t)}{K_1^{\theta_1}(t)}e^{\theta_1x_1(t)}dt\\&+\int_0^{\omega}r_1(t)\alpha_{12}(t)\frac{e^{x_2(t)}}{K_2(t)}dt\end{aligned} \tag{5}$$

$$\begin{aligned}\overline{r}_2\omega&=-\sum_{k=1}^{q}\ln(1+p_k^2)+\int_0^{\omega}\frac{r_2(t)}{K_2^{\theta_2}(t)}e^{\theta_2x_2(t)}dt\\&+\int_0^{\omega}r_2(t)\alpha_{21}(t)\frac{e^{x_1(t)}}{K_1(t)}dt\end{aligned} \tag{6}$$

From (4)- (6), we obtain

$$\int_0^{\omega}|x_1'(t)|dt\le 2\overline{r}_1\omega+\sum_{k=1}^{q}\ln(1+p_{1k})\equiv A_1 \tag{7}$$

$$\int_0^{\omega}|x_2'(t)|dt\le 2\overline{r}_2\omega+\sum_{k=1}^{q}\ln(1+p_{2k})\equiv A_2 \tag{8}$$

Since $x_i(t)\in PC([0,\omega],\mathbb{R})$, there exist $\xi_i,\eta_i\in[0,\omega]$, such that

$$x_i(\xi_i)=\inf_{t\in[0,\omega]}x_i(t),$$
$$x_i(\eta_i)=\sup_{t\in[0,\omega]}x_i(t),\ i=1,2.$$

It follows from (5) that

$$\begin{aligned}\frac{1}{(K_1^M)^{\theta_1}}\overline{r}_1\omega e^{\theta_1x_1(\xi_1)}&\le\int_0^{\omega}\frac{r_1(t)}{K_1^{\theta_1}(t)}e^{\theta_1x_1(t)}dt\\&\le\overline{r}_1\omega+\sum_{k=1}^{q}\ln(1+p_{1k})\end{aligned}.$$

which implies

$$x_1(\xi_1)\le\frac{1}{\theta_1}\ln[1+\frac{1}{\overline{r}_1\omega}\sum_{k=1}^{q}\ln(1+p_{1k})]+\ln K_1^M\equiv B_1.$$

Thus we get

$$\begin{aligned}x_1(t)&\le x_1(\xi_1)+\int_0^{\omega}|x_1'(t)|dt+\sum_{k=1}^{q}|\ln(1+\mathrm{p}_{1k})|\\&\le B_1+A_1+\sum_{k=1}^{q}|\ln(1+\mathrm{p}_{1k})\equiv C_1\end{aligned} \tag{9}$$

In particular, we have $x_1(\eta_1)\le C_1$.

Similarly, there exists a constant

$$B_2=\frac{1}{\theta_2}\ln[1+\frac{1}{\overline{r_2}\omega}\sum_{k=1}^{q}\ln(1+p_{2k})]+\ln K_2^M$$

such that

$$x_2(t)\le B_2+A_2+\sum_{k=1}^{q}|\ln(1+p_{2k})|\equiv C_2 \qquad (10)$$

On the other hand, from (5), we have

$$\overline{r_1}\omega\le-\sum_{k=1}^{q}\ln(1+p_{1k})+\frac{\overline{r_1}\omega}{(K_1^L)^{\theta_1}}e^{\theta_1 x_1(\eta_1)} +\frac{1}{K_2^L}\overline{r_1\alpha_{12}}\omega e^{x_2(\eta_2)}.$$

Then we get

$$\frac{\overline{r_1}\omega}{(K_1^L)^{\theta_1}}e^{\theta_1 x_1(\eta_1)}\ge\sum_{k=1}^{q}\ln(1+p_{1k})+\overline{r_1}\omega-\frac{1}{K_2^L}\overline{r_1\alpha_{12}}\omega e^{C_2}.$$

Because of (H3) we have

$$x_1(\eta_1)\ge\frac{\ln\left(\sum_{k=1}^{q}\ln(1+p_{1k})+\overline{r_1}\omega-\frac{1}{K_2^L}\overline{r_1\alpha_{12}}\omega e^{C_2}\right)}{\theta_1} +\ln(K_1^L)-\frac{\ln(\overline{r_1}\omega)}{\theta_1}\equiv D_1.$$

Thus we get

$$\begin{aligned}x_1(t)&\ge x_1(\eta_1)-\int_0^{\omega}|x_1'(t)|\,dt-\sum_{k=1}^{q}|\ln(1+p_{1k})|\\&\ge D_1-A_1-\sum_{k=1}^{q}|\ln(1+p_{1k})|\equiv E_1\end{aligned} \qquad (11)$$

For the same reason, there exists a constant

$$D_2=\frac{\ln\left(\sum_{k=1}^{q}\ln(1+p_{2k})+\overline{r_2}\omega-\frac{1}{K_1^L}\overline{r_2\alpha_{21}}\omega e^{C_1}\right)}{\theta_2} +\ln(K_2^L)-\frac{\ln(\overline{r_2}\omega)}{\theta_2},$$

such that

$$x_2(t)\ge D_2-A_2-\sum_{k=1}^{q}|\ln(1+p_{2k})|\equiv E_2. \qquad (12)$$

From (9)-(12), it follows that

$$|x_1(t)|\le F_1=\max\{|C_1|,|E_1|\},$$
$$|x_2(t)|\le F_2=\max\{|C_2|,|E_2|\}.$$

Obviously, F_1 and F_2 are independent of λ. Thus, there exists a constant $F_3>0$, such that $\max\{|x_1|,|x_2|\}\le F_3$. Let

$$r>F_1+F_2+F_3,$$
$$\Omega=\{x\in X:\|x\|_c<r\},$$

then it is clear that Ω satisfies condition (a) of the Lemma 1 and N is L-compact on $\bar{\Omega}$.

When

$$x=(x_1,x_2)^T\in\partial\Omega\cap\ker L=\partial\Omega\cap\mathbb{R}^2,$$

x is a constant vector in $\mathbb{R}^2$ with $\|x\|=r$, thus

$$QNx=\left(\begin{pmatrix}\overline{r_1}-\frac{1}{\omega}e^{\theta_1x_1}\int_0^{\omega}\frac{r_1(t)}{K_1^{\theta_1}(t)}dt\\ -\frac{1}{\omega}e^{x_2}\int_0^{\omega}\frac{\alpha_{12}(t)}{K_2(t)}r_1(t)dt\\ +\frac{1}{\omega}\sum_{k=1}^{q}(1+p_{1k})\\ \overline{r_2}-\frac{1}{\omega}e^{\theta_2x_2}\int_0^{\omega}\frac{r_2(t)}{K_2^{\theta_2}(t)}dt\\ -\frac{1}{\omega}e^{x_1}\int_0^{\omega}\frac{\alpha_{21}(t)}{K_1(t)}r_2(t)dt\\ +\frac{1}{\omega}\sum_{k=1}^{q}(1+p_{2k})\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}_{2\times q}\right)\ne0$$

Let $J:\operatorname{Im}Q\to\ker L$, $(d,0,\ldots,0)\to d$. A direct computation gives $\deg\{JQN,\Omega\cap\ker L,0\}\ne0$. By now we have proved that Ω satisfies all the requirements in Mawhin's continuation theorem. Hence, (3) has at least one ω-periodic solution. By (2), we derive that (1) has at least one positive ω-periodic solution. The proof is complete.

III. AN ILLUSTRATIVE EXAMPLE

In system (1), we take (H5):

$t_k=kT$, $r_1(t)=5+0.6\sin t$, $r_2(t)=4+0.4\cos t$,

$K_1(t)=3+0.3\sin t$, $K_2(t)=3+0.2\sin t$, $\theta_1=1.5$,

$\theta_2=1.6$, $\alpha_{12}(t)=0.8+0.1\cos t$,

$\alpha_{21}(t)=0.9+0.2\sin t$. $p_{1k}=0.3$, $p_{2k}=0.5$.

Obviously, $r_1(t)$, $r_2(t)$, $K_1(t)$, $K_2(t)$, α_{12}, α_{21}, satisfy (H1).

If $T=2\pi/3$, then system (1) under the conditions (H5) has a unique 2π-periodic solution (see Fig.1-Fig.3, we take $(y_1(0),y_2(0))^T=(0.5,0.5)^T$). The influence of the period pulses is obvious.

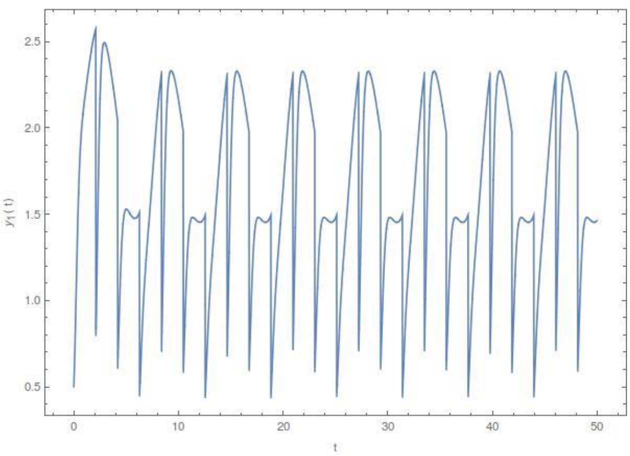

Figure 1. Time-series of $y_1(t)$ evolved in system (1) with $T=2\pi/3$.

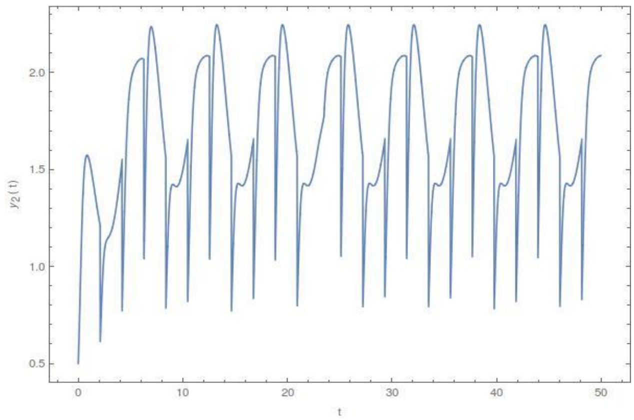

Figure 2. Time-series of $y_2(t)$ evolved in system (1) with $T = 2\pi/3$.

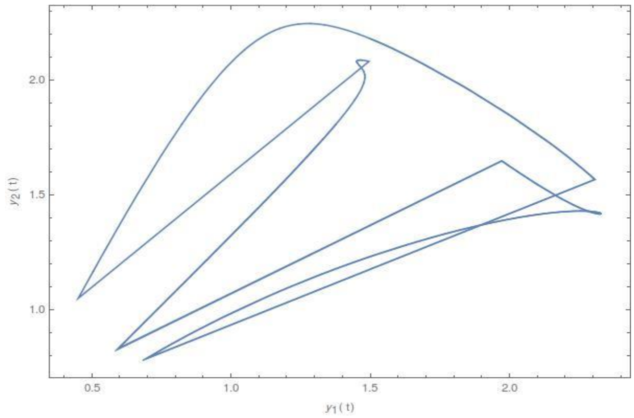

Figure 3. Phase portrait of 2π -periodic solutions of system (1) with $T = 2\pi/3$

However, if $T = 2$, then (H2) is not satisfied. Periodic oscillation of system (1) under the conditions (H5) will be destroyed by impulsive effect. Numeric results show that system (1) under the conditions (H5) has Gui chaotic strange attractor (see Fig.4) [8-9]. In Fig.4, we take $(y_1(0), y_2(0))^T = (0.5, 0.5)^T$.

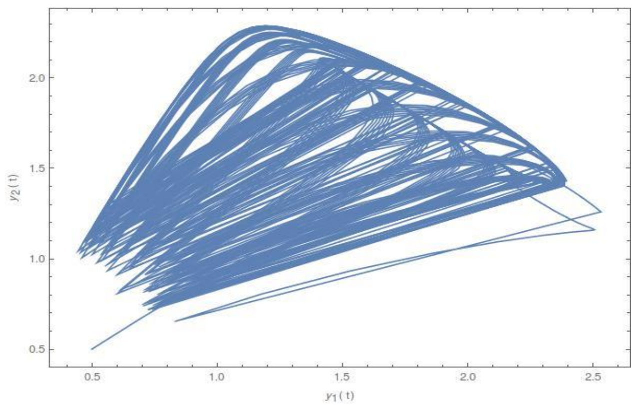

Figure 4. Phase portrait of Gui chaotic strange attractor of system (1) with $T = 2$.

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