Some issues on Distance and Similarity measures of dual hesitant fuzzy sets

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Abstract—The Dual Hesitant Fuzzy Sets (DHFSs) is a useful tool to deal with vagueness and ambiguity in the multiple attribute decision making (MADM) problems. The distance and similarity measures analysis are important research topics. In this paper, we propose some new distance measures for dual hesitant fuzzy sets, study the properties of the measurs. In the end, we develop an approach for multi-criteria decision making under dual hesitant fuzzy environment, and illustrate an example to show the behavior of the proposed distance measures.

Keywords- dual hesitant fuzzy set; distance measures; similarity measures; multi-criteria decision making

I. INTRODUCTION

Zhu and Xu [1] introduced the definition of dual hesitant fuzzy set, which is a new extension of fuzzy sets (FSs) [2]. Zhu and Xu[1] 's DHFSs used the membership hesitancy function and the non-membership hesitancy function to support a more exemplary and flexible access to assign values for each element in the domain. DHFS can be regarded as a more comprehensive set, which supports a more flexible approach when the decision makers provide their judgments. The existing sets, including FSs [2], IFSs [3] and HFSs [4] can be regarded as special cases of DHFSs.

Distance measures of FSs are an important research topic in the FS theory, which has received much attention from researchers [6-9]. Among them, the most widely used distance measures [10-12] are the Hamming distance, Euclidean distance, and Hausdorff metric. Later on, the distance measures about other extensions of fuzzy sets have also been developed. Later on, the distance measures about other extensions of fuzzy sets have also been developed [13-18,]. The aforementioned measures, however, cannot be used to deal with the distance measures of dual hesitant fuzzy information. However, little has been done about this issue. Thus it is very necessary to develop some theories about dual hesitant fuzzy sets. To do this, the remainder of the paper is organized as follows. Section 2 presents some basic concepts related to IFSs, HFSs and DHFSs. Section 3 aims to present the axioms for distance measures, gives some new distance measures for DHFSs. In Section 4, proposes an approach to multi-criteria decision making. Section 5 gives some conclusions.

II. PRELIMINARIES

In intuitionistic fuzzy environments, the most widely used distance measures for two IFSs A and B on $X=\{x_1,x_2,\mathrm{K},x_n\}$ are the following [13]:

the normalized Hamming distance:

$$d_{nh}(A,B)=\frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix}|\mu_A(x_i)-\mu_B(x_i)|\\+|\nu_A(x_i)-\nu_B(x_i)|\end{pmatrix}; \quad (1)$$

the normalized Euclidean distance:

$$d_{ne}(A,B)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix}(\mu_A(x_i)-\mu_B(x_i)^2)+\\(\nu_A(x_i)-\nu_B(x_i)^2)\end{pmatrix}}; \quad (2)$$

the Hausdorff metric:

$$d_{hd}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\begin{Bmatrix}|\mu_A(x_i)-\mu_B(x_i)|,\\|\nu_A(x_i)-\nu_B(x_i)|\end{Bmatrix} \quad (3)$$

A hesitant fuzzy set, allowing the membership of an element to be a set of several possible values, is very useful to express people's hesitancy in daily life. Xu and Xia [17] proposed a variety of distance measures for hesitant fuzzy sets.

$$d_{gnh}(A,B)=\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\begin{vmatrix}h_A^{\sigma(j)}(x_j)\\-h_B^{\sigma(j)}(x_j)\end{vmatrix}^{\lambda}\right)\right]^{1/\lambda}, \text{where } \lambda>0; \quad (4)$$

Where $l_{x_i} = \max\{l(h_A(x_i)), l(h_B(x_i))\}$ for each x_i in X, $l(h_A(x_i))$ *and* $l(h_B(x_i))$ represent the number of values in $h_A(x_i)$ *and* $h_B(x_i)$,respectively. We will talk about l_{x_i} in detail in the next section.Wherever Times is specified, Times Roman or Times New Roman may be used. If neither is available on your word processor, please use the font closest in appearance to Times. Avoid using bit-mapped fonts if possible. True-Type 1 or Open Type fonts are preferred. Please embed symbol fonts, as well, for math, etc.

III. DISTANCE AND SIMILARITY MEASURES FOR DHFES

A lot of distance and similarity measures have been developed for FSs, IFSs and HFSs [14-18], but there is little research on DHFSs. Consequently, it is very necessary to develop some distance and similarity measures under dual hesitant fuzzy environment. We first address this issue by putting forward the axioms for distance and similarity measures.

Definition 1 [1]. Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) D on X is described as:

$$D = \{< x, h(x), g(x) >| x \in X\}$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D respectively, with the conditions:

$$0 \le g, h \le 1, 0 \le g^+ + h^+ \le 1$$

where

$g \in h(x)$, $h \in g(x)$, $g^+ \in h^+(x) = \bigcup_{g \in h(x)} \max\{g\}$, and $h^+ \in g^+(x) = \bigcup_{h \in g(x)} \max\{h\}$ for all $x \in X$. For convenience, the pair $d_E(x) = (h_E(x), g_E(x))$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = (h, g)$.

Definition 2. Let A and B be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the distance between A and B denoted as $d(A,B)$, which satisfy the following properties:

1) $0 \le d(A,B) \le 1$;

2) $d(A,B) = 0$ *if only if* $A = B$;

3) $d(A,B) = d(B,A)$.

Definition 3. Let A and B be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the similarity measure between A and B is defined as $s(A,B)$, which satisfy the following properties:

1) $0 \le s(A,B) \le 1$;

2) $s(A,B) = 1$ *if only if* $A = B$;

3) $s(A,B) = s(B,A)$.

By analyzing Definitions 2 and 3, we can see the higher the similarity is, the smaller the distance between the two DHFEs. It is noted that $s(A,B) = 1 - d(A,B)$. Accordingly, we mainly discuss the distance measures for DHFSs in this paper, and the corresponding similarity measures can be obtained easily.

We arrange the elements in $d_E(x) = (h_E(x), g_E(x))$ in decreasing order, and let $\gamma_E^{\sigma(i)}(x)$ be the ith largest value in $h_E(x)$ and $\eta_E^{\sigma(j)}(x)$ be the jth largest value in $g_E(x)$. Let $l_h(d_E(x_i))$ be the number of values in $h_E(x_i)$ and $l_g(d_E(x_i))$ be the number of values in $g_E(x_i)$. For convenience, $l(d(x_i)) = \left(l_h(d(x_i)), l_g(d(x_i))\right)$. In most cases, $l(d_A(x_i)) \ne l(d_B(x_i))$, i.e., $l_h(d_A(x_i)) \ne l_h(d_B(x_i))$, $l_g(d_A(x_i)) \ne l_g(d_B(x_i))$. To operate correctly, we should extend the shorter one until both of them have the same length when we compare them. The same situation can also be found in many existing Refs [17,19, 20].

We develop a generalized hybrid dual hesitant weighted distance combining the generalized dual hesitant weighted distance and the generalized dual hesitant weighted Hausdorff distance as:

$$d_1(A,B) = \sum_{i=1}^{n} \left(w_i \left(\begin{array}{l} \dfrac{1}{2l_{x_i}} \left(\sum_{j=1}^{\#h_{x_i}} \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|^\lambda + \sum_{k=1}^{\#g_{x_i}} \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right|^\lambda \right) \\ + \dfrac{\max\left\{\max_j \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|^\lambda, \max_k \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right|^\lambda\right\}}{2} \end{array} \right) \right)^{1/\lambda};$$

where $\lambda > 0$. (5)

We find that the generalized dual hybrid hesitant weighted distance are one fundamental distance measure, based on which all of the other developed distance measures can be obtained under some special conditions.

Furth more, Motivated by the ordered weighted idea [17, 21-23], we defined a generalized dual hesitant ordered weighted distance measure:

$$d_2(A,B) = \sum_{i=1}^{n} \left(w_i \left(\frac{1}{l_{x_{\hat\sigma(i)}}} \left(\begin{array}{l} \sum_{j=1}^{\#h_{x_{\hat\sigma(i)}}} \left|\gamma_A^{\sigma(j)}(x_{\hat\sigma(i)}) - \gamma_B^{\sigma(j)}(x_{\hat\sigma(i)})\right|^\lambda \\ + \sum_{k=1}^{\#g_{x_{\hat\sigma(i)}}} \left|\eta_A^{\sigma(k)}(x_{\hat\sigma(i)}) - \eta_B^{\sigma(k)}(x_{\hat\sigma(i)})\right|^\lambda \end{array} \right) \right) \right)^{1/\lambda};$$

Where $\lambda > 0$, (6)

And $\hat\sigma : (1,2,3,\ldots n) \to (1,2,3,\ldots n)$ be a permutation satisfying:

$$\frac{1}{l_{x_{\hat\sigma(i)}}} \left(\begin{array}{l} \sum_{j=1}^{\#h_{x_{\hat\sigma(i)}}} \left|\gamma_A^{\sigma(j)}(x_{\hat\sigma(i)}) - \gamma_B^{\sigma(j)}(x_{\hat\sigma(i)})\right|^\lambda \\ + \sum_{k=1}^{\#g_{x_{\hat\sigma(i)}}} \left|\eta_A^{\sigma(k)}(x_{\hat\sigma(i)}) - \eta_B^{\sigma(k)}(x_{\hat\sigma(i)})\right|^\lambda \end{array} \right)$$

$$\le \frac{1}{l_{x_{\hat\sigma(i+1)}}} \left(\begin{array}{l} \sum_{j=1}^{\#h_{x_{\hat\sigma(i+1)}}} \left|\gamma_A^{\sigma(j)}(x_{\hat\sigma(i+1)}) - \gamma_B^{\sigma(j)}(x_{\hat\sigma(i+1)})\right|^\lambda \\ + \sum_{k=1}^{\#g_{x_{\hat\sigma(i+1)}}} \left|\eta_A^{\sigma(k)}(x_{\hat\sigma(i+1)}) - \eta_B^{\sigma(k)}(x_{\hat\sigma(i+1)})\right|^\lambda \end{array} \right),$$

$i = 1, 2, \ldots n-1$

Another important issue is the determination of the weight vectors associated with the ordered weighted

distance measures. Inspired by Xu and Xia [17,18], below we give three ways to determine the weight vectors. Considering each element in A and B as a special DHFS, $d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)})), i = 1, 2, \ldots n$ as given above, and denoting $\hat{\sigma}, \&\&\ and\ \&\&$ as ρ, we have

(1)Let

$$w_i = \frac{d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)}))}{\sum_{k=1}^{n} d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)}))}, i = 1, 2, \ldots n, \quad (7)$$

$$then\ w_{i+1} \geq w_i \geq 0, i = 1, 2, \ldots n-1, and\ \sum_{i=1}^{n} w_i = 1$$

(2)Let

$$w_i = \frac{e^{-d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)}))}}{\sum_{k=1}^{n} e^{-d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)}))}}, i = 1, 2, \ldots n, \quad (8)$$

$$then\ w_{i+1} \geq w_i \geq 0, i = 1, 2, \ldots n-1, and\ \sum_{i=1}^{n} w_i = 1$$

(3)Let

$$\bar{d}(d_A, d_B) = \frac{1}{n}\sum_{k=1}^{n} d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)})),$$

$$\&\&(d(d_A(x_{\rho(i)}), d_B(x_{\rho(k)})), \bar{d}(d_A, d_B))$$

$$= \left| \begin{array}{l} d(d_A(x_{\rho(i)}), d_B(x_{\rho(k)})) \\ -\frac{1}{n}\sum_{k=1}^{n} d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)})) \end{array} \right|,$$

then we define

$$w_i = \frac{1 - \&\&(\bar{d}(d_A, d_B), d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)})))}{\sum_{k=1}^{n}\left(1 - \&\&(\bar{d}(d_A, d_B), d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)})))\right)}$$

$$= \frac{1 - \left| d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)})) - \frac{1}{n}\sum_{k=1}^{n} d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)})) \right|}{\sum_{k=1}^{n}\left(1 - \left| d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)})) - \frac{1}{n}\sum_{k=1}^{n} d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)})) \right|\right)},$$

$$i = 1, 2, \ldots n\ from\ which\ we\ get\ w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \quad (9)$$

We find that the weight vector derived from (7) is a monotonic decreasing sequence, the weight vector derived from (6) is a monotonic increasing sequence, and the weight vector derived from (9) combine the above two cases, i.e., the closer the value $d(d_A(x_{\rho(i)}), d_B(x_{\rho(i)}))$ to the mean $\frac{1}{n}\sum_{k=1}^{n} d(d_A(x_{\rho(k)}), d_B(x_{\rho(k)}))$, the larger the weight w_i. In the aforementioned example, if the attribute weight vector is unknown, then we can use the ordered weighted distance measures to calculate the distance between each alternative and the ideal alternative.

IV. APPROACHES TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING WITH DUAL HESITANT FUZZY INFORMATION

As an illustrative example, consider the air defense of a naval battle group. Four missiles (alternatives) $x_i\ (i = 1, 2, 3, 4)$ remain on the candidate list. Three expert teams $e_k\ (k = 1, 2, 3)$ to act as decision makers, whose weight vector is $w = (0.3, 0.3, 0.4)^T$. Four attributes are under consideration: (1) Basic capabilities, (G_1); (2) Operational capabilities (G_2) (3) Costs and technical effects (G_3) (4) Reliability (G_4). We choose a "perfect" missile which its four criteria are perfect. The weight vector of the attributes $G_j\ (j = 1, 2, 3, 4)$ is $w = (0.15, 0.20, 0.20, 0.45)^T$. The experts $e_k\ (k = 1, 2, 3)$ evaluate the missiles (alternatives) $x_i\ (i = 1, 2, 3, 4)$ with respect to the attributes $G_j\ (j = 1, 2, 3, 4)$, and construct the following three dual hesitant fuzzy decision matrices $A_k = \left(a_{ij}^{(k)}\right)_{4\times4} (k = 1, 2, 3)$ (see Table 1). Among the considered attributes, $G_1\ and\ G_2$ are the cost attributes, $G_3\ and\ G_4$ are the benefit attributes. We transform the attribute values of cost type into the attribute values of benefit type, then $A_k = \left(a_{ij}^{(k)}\right)_{4\times4} (k = 1, 2, 3)$ are transformed into $R_k = \left(r_{ij}^{(k)}\right)_{4\times4} (k = 1, 2, 3)$ (see Tables 1) To obtain the ranking of the alternatives, we improve the method of Xu [22] and Wang [25] (see Table 2).

We now present a figure to clearly demonstrate how the score values vary as the parameter λ increases and the aggregation arguments are kept fixed (see Fig .1.).

TABLE I. DUAL FUZZY DECISION MATRICE R

	G_1	G_2	G_3	G_4
A_1	{{0.5,0.4,0.3}, {0.4,0.2}}	{{0.6,0.5}, {0.3,0.2,0.1}}	{{0.6,0.4,0.3}, {0.4,0.2,0.1}}	{{0.6}, {0.4}}
A_2	{{0.8,0.7,0.6}, {0.2,0.1}}	{{0.7,0.6}, {0.3,0.2,0.1}}	{{0.7,0.6,0.5}, {0.3,0.2,0.1}}	{{0.7}, {0.2}}
A_3	{{0.9,0.8,0.7}, {0.1,0.0}}	{{0.8,0.7}, {0.2,0.1,0.0}}	{{0.8,0.7,0.6}, {0.2,0.1,0.0}}	{{0. 9}, {0.1}}
A_4	{{0.4,0.3,0.1}, {0.6,0.5}}	{{0.6,0.5}, {0.4,0.2,0.1}}	{{0.6,0.5,0.4}, {0.3,0.2,0.1}}	{{0.3}, {0.6}}

TABLE II. SCORE VALUES OBTAINED BY ATS-WGDHFPA OPERATOR BASED ON THE GENERALIZED DUAL HESITANT WEIGHTED HAUSDORFF DISTANCE AND THE RANKING OF ALTERNATIVES.

d_1	λ=0.05	λ=0.1	λ=1	λ=10	λ=100
A_1	0.2104	0.2107	0.2157	0.2675	0.1032
A_2	0.4831	0.4833	0.4864	0.5207	0.5530
A_3	0.6528	0.6529	0.6551	0.6784	0.7081
A_4	-0.0661	-0.0658	-0.0604	-0.0115	-0.5224

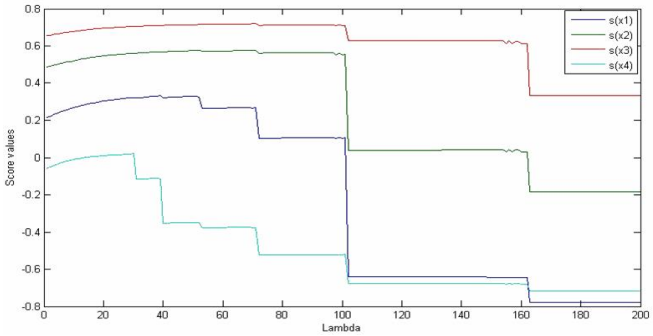

Figure 1. Variation of the score valuse with respect to the parameter λ.

V. CONCLUSION

In this paper, we have given a further study about the distance measures for DHFSs. Based on ideas of the well-known Hamming distance, the Euclidean distance, the Hausdorff metric and their generalizations, we have developed some new dual hesitant distance measures, and discussed their properties and relations as their parameters change. We have also given the ordered weighted distance measures for DHFSs. An approach for multi-criteria decision making has been developed based on the proposed distance measures under dual hesitant fuzzy environments.

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