The Existence and Simulations of Periodic Solution of Predator-Prey Models with Beddington-DeAngelis Functional Response and Impulsive Perturbations

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Abstract—In this paper, a predator-prey system with Beddington-DeAngelis functional response is studied, where the n predators prey on one prey. Using the continuation theorem of coincidence degree theory and analysis techniques, a criteria for the existence of periodic solutions of predator-prey models with the Beddington-DeAngelis functional response governed by impulsive differential equations is established. Further, some numerical simulations show that our models can occur in many forms of complexities including periodic oscillation and Gui chaotic strange attractor.

Keywordscomponent Periodic-Solution; Predator-Prey Model; Coincidence degree theory; Impulses; Beddington-DeAngelis Functional Response.

I. INTRODUCTION

Recently, many authors have devoted their efforts to the predator-prey system with Beddington-DeAngelis functional response which was introduced by Beddington and DeAngelis et al. independently (see [1-2]). It is well known that the traditional predator-prey systems with prey-dependent functional response fail to model the interference among predators. To overcome this shortcoming, Arditi and Ginzburg [3] proposed the ratio-dependent predator-prey model which incorporates mutual interference by predators.

Although much progress has been seen in the study of predator-prey models with the Beddington-DeAngelis functional response (eg. [4,5]), such models are not well studied yet in the sense that all the existing results are based on the assumption that the predator preys on one prey. This assumption is rarely the case in real life. Naturally, more realistic and interesting model should take into account the predator preying on more than one prey. Therefore, Z.J. Zeng and M. Fan [6] established a more reasonable model with multiple preys. However, the corresponding problems (eg. [7]) with periodic coefficients and impulsive perturbations are studied far less often. In this paper, we will consider the following system:

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= x(t)\left[a(t)-b(t)x(t) - \sum_{i=1}^{n}\frac{c_i(t)y_i(t)}{\alpha_i(t)+\beta_i(t)x(t)+\gamma_i(t)y_i(t)}\right] \\ \frac{dy_i}{dt} &= y_i(t)\left[-d_i(t) + \frac{f_i(t)x(t)}{\alpha_i(t)+\beta_i(t)x(t)+\gamma_i(t)y_i(t)}\right] \end{aligned}\right\} t\neq t_k, \\ \left.\begin{aligned} \Delta x(t_k) &= p_k x(t_k) \\ \Delta y_i(t_k) &= q_k^i y_i(t_k) \end{aligned}\right\} t=t_k, \end{cases} \tag{1}$$

where $x(t)$ and $y_i(t)$ ($i=1,2,\ldots,n$) represent prey and predator densities respectively. $a(t)$ stands for prey intrinsic growth rate; $d_i(t)$ stands for the death rate of the predator; $c_i(t)$ and $f_i(t)$ are the uptake and predation constants of the predator and prey; $\alpha_i(t),\beta_i(t),\gamma_i(t)$ are positive constants; p_k, q_k^i are constant.

In system (1), we assume:

(H1) $c_i(t),a(t),b(t),d_i(t),\alpha_i(t),\beta_i(t),\gamma_i(t)$ and $f_i(t)$ are all positive periodic continuous functions with period $\omega>0$.

(H2) There exist a positive constant M and constant N_i which satisfy $0\le N_i\le 1$ such that $p_k\le M$, $q_k^i\le N_i$, $k=1,2.$.

(H3) $I_k=1+p_k>0$, $\overline{I}_{ik}=1+q_k^i>0$ and there exists a positive integer q such that $t_{k+q}=t_k+\omega$, $q_{k+q}^i=q_k^i$, $p_{k+q}=p_k$, $(i=1,2,...,n)$.

II. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

To prove our results, we need the notion of the Mawhin's continuation theorem formulated in [8].

Lemma 1 Let X and Y be two Banach spaces. Consider an operator equation $Lx=\lambda Nx$ where L: $\operatorname{Dom} L \cap X \to Y$ is a Fredholm operator of index zero and $\lambda \in [0,1]$ is a parameter. Let P and Q denote two projectors such that $P: X \to \operatorname{Ker} L$ and $Q: Y \to Y/\operatorname{Im} L$. Assume that $N:\bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$, where Ω is open bounded in X. Furthermore, assume that

(a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;

(b) for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;

(c) $\deg\{JQNx, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $J: \operatorname{Im} Q \to \operatorname{Ker} L$ and $\deg\{*\}$ represents the Brouwer degree.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.

For the sake of convenience, we introduce the following notations:

$$\bar{u} = \frac{1}{\omega}\int_0^{\omega} u(t)dt, \quad g_i^{\,l} = \min_{t\in[0,\omega]} g_i(t)$$
$$g_i^{\,u} = \max_{t\in[0,\omega]} g_i(t), \quad (i=1,2,...,n)$$

$$PC(J,\mathbb{R}) = \left\{ \begin{array}{l} x: J \to \mathbb{R} \,\big|\, x(t) \text{ is continuous} \\ \quad \text{with respect to } t \neq t_1,...,t_q; \\ \quad x(t^+) \text{ and } x(t^-) \text{ exsit at } t_1,...,t_q; \\ \quad \text{and } x(t_k)=x(t_k^+), k=1,2,...,\mathrm{q} \end{array} \right\}$$

Lemma 2 $x^*(t)$ is an ω-periodic solution of (1) if and only if $\ln\{x^*(t)\}$ is an ω-periodic solution of

$$\begin{cases} \left.\begin{array}{l} \dfrac{dx}{dt} = a(t) - b(t)\exp\{x(t)\} \\ \qquad -\displaystyle\sum_{i=1}^{n} \frac{c_i(t)\exp\{y_i(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}} \\ \dfrac{dy_i}{dt} = -d_i(t) \\ \qquad + \dfrac{f_i(t)\exp\{x(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}} \end{array}\right\} t \neq t_k, \\ \left.\begin{array}{l} \Delta x(t_k) = I_k \\ \Delta y_i(t_k) = \bar{I}_{ik} \end{array}\right\} t = t_k. \end{cases} \tag{2}$$

Now we are ready to state and prove the main results of the present paper.

Theorem 1 Assume (H1)-(H3) hold, furthermore, we assume:

(H4) $\left(\bar{f}_i\omega - \beta_i^{\,u}(\bar{d}_i\omega - p\ln(1-N_i))\right)\exp\{M_1\}$
$\quad > \alpha_i^{\,u}(\bar{d}_i\omega - p\ln(1-N_i))$

(H5) $\bar{a} - \displaystyle\sum_{i=1}^{n}\overline{c_i/\gamma_i} > 0$,

where $M_1 = \ln \dfrac{1}{\bar{b}\exp\{2\bar{a}\omega\}(1+M)^{3p}}\left(\bar{a} - \displaystyle\sum_{i=1}^{n}\overline{c_i/\gamma_i}\right)$, $(i=1,2...,n)$.

Then system (1) has at least one ω-periodic solution.

Proof. To complete the proof, we only need to search for an appropriate open bounded subset $\Omega \subset X$ verifying all the requirements in Lemma 1.

Note $u = (x(t), y_1(t), ..., y_n(t))^T$, obviously $I_k \geq 0$, $\bar{I}_{ik} \leq 0$. Define $X = \{u \in PC(\mathbb{R}, \mathbb{R}^{n+1}) \,|\, u(t+\omega) = u(t)\}$, $Y = X \times \mathbb{R}^{2p}$, then it is standard to show that both X and Y are Banach space when they are endowed with the norms $\|u\|_c = \sup_{t\in[0,\omega]}|x(t)|$ and $\|(u,c_1,...,c_p)\| = (\|u\|_p^2 + |c_1|^2 + ... + |c_p|^2)^{1/2}$.

Let $\operatorname{Dom} L = X = \{u \in C^1[0,\omega,t_1,...,t_p] \,|\, u(0) = u(\omega)\}$, $L: \operatorname{Dom} L \to Y$, $Lx = (u', \Delta u(t_1), ..., \Delta u(t_p))$,

$$Nx \left(\begin{pmatrix} a(t) - b(t)\exp\{x(t)\} \\ -\displaystyle\sum_{i=1}^{n} \frac{c_i(t)\exp\{y_i(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}} \\ -d_i(t) \\ +\displaystyle\sum_{i=1}^{n} \frac{f_i(t)\exp\{x(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}} \end{pmatrix}, \begin{pmatrix} I_1,...,I_p \\ \bar{I}_{i1},...,\bar{I}_{ip} \end{pmatrix} \right)$$

It is easy to prove that L is a Fredholm mapping of index zero.

Consider the operator equation

$$Lx = \lambda Nx \quad \lambda \in (0,1). \tag{3}$$

Integrating (4) over the interval $[0,\omega]$, we obtain

$$\begin{aligned} \bar{a}\omega &= -\sum_{k=1}^{p} I_k + \int_0^{\omega} b(t)\exp\{x(t)\}dt \\ &\quad + \int_0^{\omega}\sum_{i=1}^{n}\frac{c_i(t)\exp\{y_i(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}}dt, \end{aligned} \tag{4}$$

$$\begin{aligned} \bar{d}_i\omega &= \sum_{k=1}^{p}\bar{I}_{ik} \\ &\quad + \int_0^{\omega}\frac{f_i(t)\exp\{x(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}}dt, \\ & i=1,2,...,n. \end{aligned} \tag{5}$$

We can derive

$$\int_0^{\omega}|x'(t)|dt \leq 2\bar{a}\omega + 2p\ln(1+M),$$

$$\begin{aligned} \int_0^{\omega}|y_i'(t)|dt &\leq \int_0^{\omega}|d_i(t)|dt \\ &\quad + \int_0^{\omega}\left|\sum_{i=1}^{n}\frac{f_i(t)\exp\{x(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}}\right|dt \\ &\quad + \sum_{k=1}^{p}\bar{I}_{ik} \\ &\leq 2\bar{d}_i\omega. \end{aligned}$$

Since $x(t) \in PC([0,\omega],\mathbb{R})$, $y_i(t) \in PC([0,\omega],\mathbb{R})$, there exist $\xi, \eta, \xi_i, \eta_i \in [0,\omega] \cup [t_1^+, t_2^+, ..., t_q^+]$, such that $x(\xi) = \inf_{t\in[0,\omega]} x(t)$, $x(\eta) = \sup_{t\in[0,\omega]} x(t)$, $y_i(\xi_i) = \inf_{t\in[0,\omega]} y_i(t)$, $y_i(\eta_i) = \sup_{t\in[0,\omega]} y_i(t)$, $i=1,2,...n$

From (5), we can see

$$\bar{a}\omega \le \int_0^{\omega} b(t)\exp\{x(\eta)\}\,dt +$$

$$+\int_0^{\omega}\sum_{i=1}^{n}\frac{c_i(t)\exp\{y_i(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}}dt$$

$$\le \int_0^{\omega} b(t)\exp\{x(\eta)\}\,dt + \int_0^{\omega}\sum_{i=1}^{n}\frac{c_i(t)\exp\{y_i(t)\}}{\gamma_i(t)\exp\{y_i(t)\}}dt$$

$$\le \bar{b}\,\omega\exp\{x(\eta)\} + \sum_{i=1}^{n}\overline{c_i/\gamma_i}\,\omega.$$

It follows that

$$x(\eta) \ge \ln\frac{1}{\bar{b}}\left(\bar{a} - \sum_{i=1}^{n}\overline{c_i/\gamma_i}\right) := l_1$$

$$\int_0^{\omega} b(t)\exp\{x(\xi)\} \le \int_0^{\omega} b(t)\exp\{x(t)\} \le \bar{a}\omega + \sum_{k=1}^{p} I_k \ .$$

$$\bar{b}\,\omega\exp\{x(\xi)\} \le \bar{a}\omega + p\ln(1+M)\ .$$

It follows that

$$x(\xi) \le \ln\left[\frac{\bar{a}\omega + p\ln(1+M)}{\bar{b}\omega}\right] := L_1\ .$$

Then for $\forall t\in[0,\omega]$, we have

$$x(t) \ge x(\eta) - \sum_{k=1}^{p} I_k - \int_0^{\omega}|x'(t)|dt$$

$$\ge l_1 - 2\bar{a}\omega - 3p\ln(1+M) \ge M_1,$$

$$x(t) \le x(\xi) + \sum_{k=1}^{p} I_k + \int_0^{\omega}|x'(t)|dt$$

$$\le \ln\left[\frac{(\bar{a}\omega + p\ln(1+M))\exp\{2\bar{a}\omega\}(1+M)^{3p}}{\bar{b}\omega}\right] := M_2$$

then we can derive

$$|\,x(t)\,| \le \max\{\,|\,M_1\,|,|\,M_2\,|\,\} := N_1$$

Similarly, according to (4), for $\forall t\in[0,\omega]$, we have

$$\bar{d}_i\omega \le \int_0^{\omega}\frac{f_i(t)\exp\{x(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}}dt$$

$$\le \int_0^{\omega}\frac{f_i(t)\exp\{M_2\}}{\gamma_i^{\,l}\exp\{y_i(\xi_i)\}}dt \le \frac{\bar{f}_i\omega\exp\{M_2\}}{\gamma_i^{\,l}\exp\{y_i(\xi_i)\}}.$$

$$y_i(\xi_i) \le \ln\left[\frac{\bar{f}_i\exp\{M_2\}}{\bar{d}_i\gamma_i^{\,l}}\right] := C_{i1}\ .$$

So, for $\forall t\in[0,\omega]$, we have

$$y_i(t) \le y_i(\xi_i) - \sum_{k=1}^{p}\bar{I}_{ik} + \int_0^{\omega}|y_i'(t)|dt \quad .$$

$$\le C_{i2} - p\ln(1-N_i) + 2\bar{d}_i\omega := M_{i3}$$

From (6), we also have

$$\bar{d}_i\omega - \sum_{k=1}^{p}\bar{I}_{ik} = \int_0^{\omega}\frac{f_i(t)\exp\{x(t)\}}{\alpha_i(t)+\beta_i(t)\exp\{x(t)\}+\gamma_i(t)\exp\{y_i(t)\}}dt$$

$$\ge \frac{\bar{f}_i\omega\exp\{M_1\}}{\alpha_i^{\,u} + \beta_i^{\,u}\exp\{M_1\} + \gamma_i^{\,u}\exp\{y_i(\eta_i)\}},$$

then we can derive

$$y_i(\eta_i) \ge \ln\left[\frac{S}{\gamma_i^{\,u}(\bar{d}_i\omega - p\ln(1-N_i))}\right] := C_{i2},$$

where

$$S = \left(\bar{f}_i\omega - \beta_i^{\,u}(\bar{d}_i\omega - p\ln(1-N_i))\right)\exp\{M_1\}$$
$$-\alpha_i^{\,u}(\bar{d}_i\omega - p\ln(1-N_i)).$$

Thus

$$y_i(t) \ge y_i(\eta_i) + \sum_{k=1}^{p}\bar{I}_{ik} - \int_0^{\omega}|y_i'(t)|dt$$

$$\ge C_{i2} + p\ln(1-N_i) - 2\bar{d}_i\omega := M_{i4}.$$

Then we can derive

$$|\,y_i(t)\,| \le \max\{\,|M_{i3}|,|M_{i4}|\,\} := N_2$$

Obviously, there exists a constant $N_3 > 0$ such that $\max\{\,|\,\bar{x}\,|,|\,\bar{y}\,|, i=1,2,...,n\} < N_3$. Take $r > N_1 + N_2 + N_3$, $\Omega = \{x\in X : \|\,x\,\|_c < r\}$ then N is L-compact on $\bar{\Omega}$. So, for $\forall u = (\bar{x},\bar{y}_1,\bar{y}_2,...,\bar{y}_n)^T \in \partial\Omega\cap \operatorname{Ker} L$, we have $QNx \ne 0$. Let $J : \operatorname{Im} Q \to X$, $(d,0,...,0)\to d$. Then for $\forall u\in\Omega\cap\operatorname{Ker} L$, in view of the assumptions in Mawhin's continuation theorem [8], one obtains $\deg\{JQNx,\Omega\cap\operatorname{Ker} L,0\} \ne 0$. By now we have proved that Ω satisfies all the requirements in Mawhin's continuation theorem. Hence, system (3) has at least one ω-periodic solution. By Lemma 2, we derive that system (1) has at least one positive ω-periodic solution. The proof is complete.

III. An Illustrative Example

In this section, we shall discuss an example to illustrate our main results. In (1), we take $n=2, t_k = kT$,

$$a(t) = 3-\sin(t),\quad d_1(t) = 1+0.5\cos(t),$$
$$c_1(t) = 1-0.3\cos(t),\quad f_1(t) = 5+2\sin(t),$$
$$b(t) = 1+0.2\cos(t),\quad d_2(t) = 1-0.1\sin(t),$$
$$c_2(t) = 0.7+0.3\sin(t),\quad f_2(t) = 4+3\cos(t),$$
$$\alpha_1(t) = 3-0.3\sin(t),\quad \beta_1(t) = 1+0.3\cos(t),$$
$$\gamma_1(t) = 0.8-0.5\sin(t),\quad \alpha_2(t) = 2+0.2\sin(t),$$
$$\beta_2(t) = 1+0.2\sin(t),\quad \gamma_2(t) = 1+0.5\cos(t).$$

If $p_k = 0.3$, $q_k^1 = 0.2$, $q_k^2 = 0.3$, all conditions of Theorem 2 are satisfied. If $T=\pi$, then system (1) under has a unique periodic solution (see Fig .1-Fig .4). Because of the influence of the period pulses, the influence of pulse is obvious.

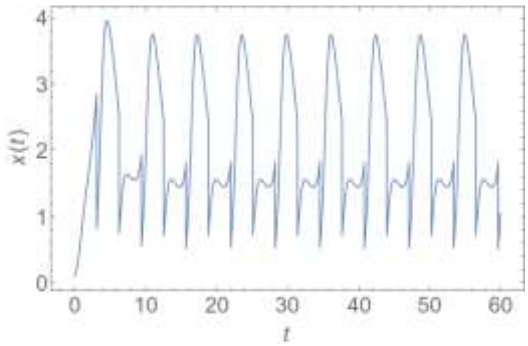

Figure 1. Time-series of $x(t)$ of system (1) with $T=\pi$.

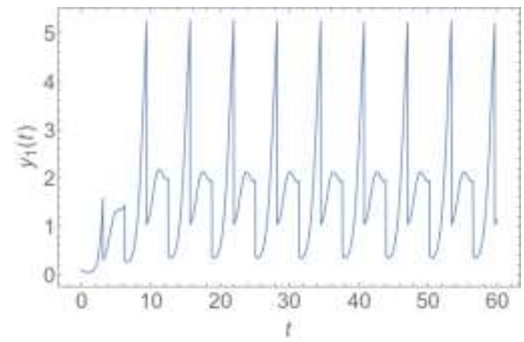

Figure 2. Time-series of $y_1(t)$ of system (1) with $T = \pi$.

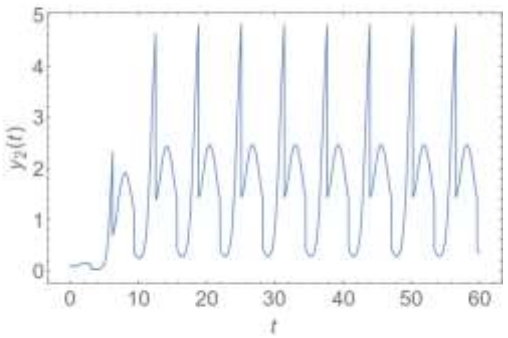

Figure 3. Time-series of $y_2(t)$ of system (1) with $T = \pi$.

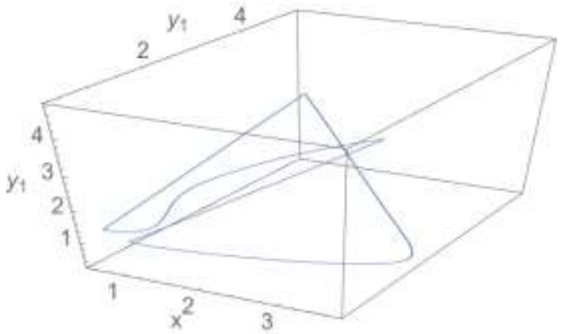

Figure 4. Phase portrait of periodic solution of system (1).

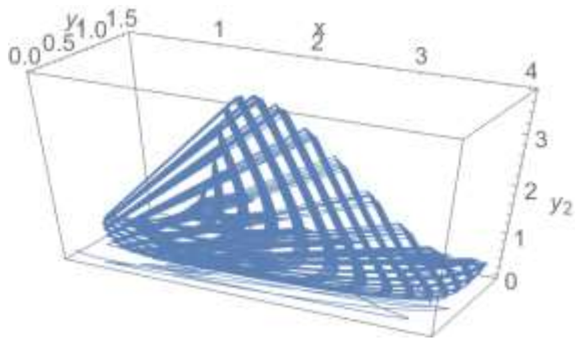

Figure 5. Phase portrait of Gui strange attractor of system (1).

If $T = 2$, then (H3) is not satisfied. Periodic oscillation of system (1) will be destroyed by impulsive effect. Numeric results show that system (1) has Gui chaotic strange attractor [9-13], see Fig .5.

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