Uncertain Engineering Critical Path Solving Method Based on Interval Number theory

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Abstract—Critical path management is an important way for controlling the schedule of an engineering project, it always difficult to describe by a precise manner for the uncertain characteristic. In this paper, the interval number is applied to describe the uncertain characteristic, and a preference indexes λ are introduced to solve the critical path on linear programming problem. Meanwhile, the control priorities of sub-path can be also analyzed by a sensitivity coefficient δ . The result shows that compared with traditional probability methods and other interval critical path studies, the proposed method can improve the precision of the project time control. From the example and the result analysis, traditional probability method can deal with the uncertain problem of the network, but it may optimistically estimate the project duration time, namely, the duration time is underestimating. But interval critical path solving method could give more accurate duration time and critical path, and the proposed method could give more exact result and the calculation procedure is simpler.

Keywords- critical path; uncertain management; engineering time control; interval number; activity management

I. INTRODUCTION

Critical path method (CPM) has been demonstrated to be a useful graphic tool in the project time control^[1-3], this technology shows that the keystone of the project control is the critical path because the activities in CPM could induce the project success or failure, and the total duration time in CPM is considered as the best times of the project^[4]. When the activity times in the project are deterministic and known, CPM has been demonstrated to be an efficient manner. However, there are many cases where the activity times may not be presented in a precise manner since the risk factor exists. To deal quantitatively with imprecise data, probability theory^[5] and Monte Carlo simulation^[6] are employed. But in these analysis, the probability distributions of each activities is needed, it is difficult to use in some situations when the priori data of the activity probability distributions are absence. Therefore, there are some detailed critiques of PERT can be found in some work^[7].

An alternative way to deal with imprecise data is to employ the fuzzy numbers. The main advantages of this methodology are that it do not require prior predictable regularities or posterior frequency distributions, and it can deal with imprecise input information based on the human Wang Jianjun School of Economics and Management North China Electric Power University Beijing, China wangjianjunhd@126.com

subjective judgment. The main idea of fuzzy numbers method^[8-9] is using -cut set into transform the fuzzy numbers to interval numbers. Therefore, Form both the theory perspective and the practice perspective, it is important to study the interval number critical path problem.

Stefan^[10] presents the definition of interval critical path (ICP) and some lemmas. But the definition is so wide that many ICPs may exist in a common ICP network, and which one is the most critical path is not discussed. And Song proposes a method based on ICP ^[11], although the most critical path is given, it adds a constraint. Therefore, a new generic interval number critical path method was proposed, and an example was solved successfully to illustrate the validity and accurate of the proposed method.

II. EXACT CPM DEFINITION

Before the ICP method was proposed, the definition of classical CPM was recalled. Assume G(V, A) denotes a directed, connected, acyclic graph network of a project, V is a finite set of nodes, and $A \subset V \times V$ is a set of arcs. t_{ij} denotes the duration time between v_i and v_j . $L = \{l_i | i = 1 L \ s\}$ is a set of paths between the start node v_1 and the end node v_n , $D = \{d_i | i = 1 L \ s\}$ is a set of the duration time of L, and the definition for the classic critical path is shown as follows:

Definition 1: if $l_k \in L$ and $d_k = \max(D)$, l_k is a critical path in G.

Based on Linear Programming(LP) method, the critical path problem is shown as in (1):

$$D(t) = \max \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij}$$

s.t.

$$\sum_{j=0}^{n} x_{1j} = 1$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki} (i = 2, K, n - 1)$$

$$\sum_{k=1}^{n} x_{kn} = 1$$

$$x_{ij} = 0, 1(i, j \in A)$$
(1)

The objective function D(t) is the maximal time between the start node and the end node. The constraints are called the flow conservation equations which indicate that flow may be neither created nor destroyed in the project network, and only one unit flow could enter the project network at the start node and leave at the end node. Since only one unit of flow could be in any arc at any one time, the variable x_{ij} must assume binary values (0 or 1) only. The critical path is the solution set of equation 1.

III. MODEL OF INTERVAL NUMBER PROBLEM

A. Interval CPM Model

From equation 1. The LP problem about the interval number graph could be acquired as in (2):

$$D = \max \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} x_{ij}$$

s.t.
$$\sum_{j=0}^{n} x_{1j} = 1$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki} (i = 2, K, n - 1)$$

$$\sum_{k=1}^{n} x_{kn} = 1$$

$$x_{ij} = 0, 1(i, j \in A)$$

(2)

Where:
$$T_{ii} = [t_{ii}, t_{ij}]$$
 is interval number

Based on interval number operation theory, the total duration time $T(D) = [\underline{T}(D), \overline{T}(D)]$ was also an interval number since each activity time was the interval number. Therefore, Equation 2 was equivalent to find the lower bound $\underline{T}(D)$ and the upper bound $\overline{T}(D)$ of the problem.

Lemma 1: $\underline{T}(D)$ is the lower bound of the graph's duration time, such that replacing the interval activity time $T_{ij} = [\underline{t}_{ij}, \overline{t}_{ij}]$ with the lower bound \underline{t}_{ij} . The LP problem is as follows:

$$\underline{T}(D) = \max \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{t}_{ij} x_{ij}$$
s.t.

$$\sum_{j=0}^{n} x_{1j} = 1$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki} (i = 2, K, n - 1)$$

$$\sum_{k=1}^{n} x_{kn} = 1$$

$$x_{ii} = 0, 1(i, j \in A)$$
(3)

Proof: Suppose the path of solution is l and the duration time is d, $\forall t_{ij} \in [\underline{t}_{ij}, \overline{t}_{ij}]$ replaces the interval activity time of the graph. If the new solution is l' and the duration is d', if l' = l, $Q[\underline{t}_{ij} \leq \overline{t}_{ij}]$, then

$$d' = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij} \ge \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij} = d ; \text{ if } l' \neq l \text{, because } l' \text{ is}$$

the critical path, from definition 1, $d' \ge d'_l \ge d$. Therefore, d is the lower bound of the graph's duration time. Similarly, the lemma 2 could also be proved.

Lemma 2: $\overline{T}(D)$ is the upper bound of the graph's duration time, such that replacing the interval activity time $T_{ij} = [\underline{t}_{ij}, \overline{t}_{ij}]$ with the upper bound \overline{t}_{ij} . The LP problem is as follows:

$$\overline{T}(D) = \max \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t}_{ij} x_{ij}$$
s.t.
$$\sum_{j=0}^{n} x_{1j} = 1$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki} (i = 2, K, n - 1)$$

$$\sum_{k=1}^{n} x_{kn} = 1$$

$$x_{ii} = 0, 1(i, j \in A)$$
(4)

B. The most ICP of the graph

In actual situation, there are many ICPs in an interval network. The key to the problem was to find the most important ICP in ICPs. Generally speaking, to solve the uncertain problem was always choosing one of the results from the result sets, in other words, one control path form the ICPs was needed to be chosen by a certain principle. Therefore, the preference index λ was defined, and the preference function was shown as follows:

$$t_{ij} = (1 - \lambda)\underline{t}_{ij} + \lambda t_{ij} = \underline{t}_{ij} + \lambda(t_{ij} - \underline{t}_{ij}) \quad (\lambda \in [0, 1])$$
(5)

Then the LP problem of the most ICP is shown as follows:

$$D = \max \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij}$$

= $\max \sum_{i=1}^{n} \sum_{j=1}^{n} [t_{ij} + \lambda(\bar{t}_{ij} - t_{ij})] x_{ij}$
s.t.
$$\sum_{j=0}^{n} x_{1j} = 1$$

(6)
$$\sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki} (i = 2, K, n - 1)$$

$$\sum_{k=1}^{n} x_{kn} = 1$$

$$x_{ii} = 0, 1(i, j \in A)$$

It was obviously to see that the lower bound of the graph's duration time was such as the solution when $\lambda = 0$, and the upper bound was the solution when $\lambda = 1$. Specially, when $\lambda = 0.5$, it's the expectation of interval

number, such as the critical path choice of the traditional probability method.

IV. EXAMPLE AND RESULT ANALYSIS

A. Example

In order to compare the method with other relative methods easily, an example is used as Fig .1.



According to equation 7, the LP problem of the example was as follows:

$$D = \max 4x_{12} + (5+\lambda)x_{13} + 4x_{24} + (3+\lambda)x_{34}$$

+7x₂₅ + (7+\lambda)x₄₇ + (3+\lambda)x₃₆ + 2x₅₈
+(3+2\lambda)x₅₇ + 7x₆₇ + (2+\lambda)x₇₈ + (2+10\lambda)x₇₉
+(3+9\lambda)x₆₉ + (18+2\lambda)x₆₁₀ + (1+5\lambda)x₈₉
+x₉₁₀ + (6+5\lambda)x₈₁₁ + (6+\lambda)x₉₁₁ + (7+4\lambda)x₁₀₁₁

 $\begin{aligned} x_{12} + x_{13} &= 1 \\ x_{12} &= x_{24} + x_{25} \\ x_{13} &= x_{34} + x_{36} \\ x_{24} + x_{34} &= x_{47} \\ x_{25} &= x_{57} + x_{58} \\ x_{36} &= x_{67} + x_{610} + x_{69} \\ x_{47} + x_{57} + x_{67} &= x_{78} + x_{79} \\ x_{58} + x_{78} &= x_{89} + x_{811} \end{aligned}$

$$x_{69} + x_{79} + x_{89} = x_{911} + x_{910}$$

 $x_{610} + x_{910} = x_{1011}$

$$\begin{aligned} x_{811} + x_{911} + x_{1011} &= 1 \\ x_{12}, x_{13}, x_{24}, x_{25}, x_{34}, x_{36}, x_{47}, x_{57}, x_{58}, \end{aligned}$$

 $\begin{aligned} x_{67}, x_{78}, x_{79}, x_{69}, x_{610}, x_{89}, x_{910}, x_{811}, \\ x_{911}, x_{1011} = 0 \text{ or } 1 \ (i, j \in A) \end{aligned} \tag{7}$

A mathematical programming software package Lingo is used to solve the above linear programs. The result is shown as Table 1(the step of λ is 0.1, $\lambda \in [0,1]$).

It is obviously to see that the preference index λ is the key point of the duration time change. In order to give the control priorities of sub-path, the risk sensitivity coefficient is defined as $\delta = \frac{dD}{d\lambda}$, then $\delta_1 = 8 < \delta_2 = 17$, It's obvious that there are more control time in D_2 . From the λ distribution of the sub-paths, the control priority of

sub-path is $v_7 \rightarrow v_9, v_{10} \rightarrow v_{11}L$, so it is effective to

control the activity time of $v_7 \rightarrow v_9$ in D_2 . Similarly, it is effective to control the activity time of $v_{10} \rightarrow v_{11}$ in D_1 .

Table I. The different ICP of the different λ

λ	D	l
0	33	1-3-6-10-11
0.1	33.8	1-3-6-10-11
0.2	34.6	1-3-6-10-11
0.3	35.4	1-3-6-10-11
0.4	36.2	1-3-6-10-11
0.5	37	1-3-6-10-11
0.6	37.8	1-3-6-10-11
0.7	38.6	1-3-6-10-11
0.8	39.4	1-3-6-10-11
0.9	40.3	1-3-4-7-9-10-11
1	42	1-3-4-7-9-10-11

B. Result analysis

From the results, the total duration time T(D) = [33, 42]. When $\lambda = 0, L$, 0.8, $l_1 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 10 \rightarrow 11\}$; When $\lambda = 0.9, 1$, $l_2 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 9 \rightarrow 10 \rightarrow 11\}$

The duration time D_1 of l_1 and the duration time D_2 of l_2 were as follows:

$$\begin{split} D_1 &= (5+\lambda) + (3+\lambda) + (18+2\lambda) + (7+4\lambda) = 33+8\lambda\\ D_2 &= (5+\lambda) + (3+\lambda) + (7+\lambda) + (2+10\lambda) + 1 + (7+4\lambda) = 25+17\lambda\\ \text{When } d_1 &= d_2, \ \lambda = \frac{8}{9} \approx 0.8889 \text{ .In other words, } \lambda = \frac{8}{9}\\ \text{was a change point of ICP, It proved to be true when Lingo was adopted, and the result was as shown in Table II. \end{split}$$

TABLE II. THE THANGE POINT OF THE CRITICAL PATH

λ	D	l
0.888	40.104	1-3-6-10-11
0.889	40.113	1-3-4-7-9-10-11

C. Comparative analysis

Table 3 is the result of the results with tranditional probability method and ICP method.

According to traditional probability PERT method, the highest expectation path is chosen as the critical path to control. It is equal to the $\lambda = 0.5$ path $l_1 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 10 \rightarrow 11\}$, the duration time is [33,41]. However, the duration time [33,42] is given by ICP method, another critical path $l_2 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 9 \rightarrow 10 \rightarrow 11\}$ is also given in the result. According to the compared analysis, the traditional probability method may estimate the project duration time optimistically, namely, the duration time is underestimate. But ICP method can give more accurate duration time and critical path.

TABLE III. THE RESULTS WITH PROBABILITY METHOD AND ICP

	Crticial path	Duration time
probability method	1-3-6-10-11	[33,41]
ICP method	1-3-6-10-11	[33,42]
	1-3-4-7-9-10-11	

Compare with DBICP method and ICP method then we can get Table IV.

	Critical path	Condition
DBICP	1-3-6-10-11	Deadline is 40
	1-3-4-7-9-10-11	Deadline is 41
ICP	1-3-6-10-11	$\lambda < \frac{8}{9}$ Deadline<40.111
	1-3-4-7-9-10-11	$\lambda > \frac{8}{9}$ Deadline>40.111

TABLE IV. THE RESULTS WITH DBICP AND ICP

According to DBICP^[13-14], the conclusion is when the deadline of the project duration time is 40 days, $l_1 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 10 \rightarrow 11\}$ is chosen; when the deadline is 41 days, $l_2 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 9 \rightarrow 10 \rightarrow 11\}$ is chosen. Similarly, ICP method also gives the result that when $\lambda \ge \frac{8}{9}$, such as the duration time is longer than 40.113 days, the ICP changed. It is obviously to see that the results is more accurate by ICP method. Otherwise, DBICP method should calculate again when the constraint time is changed. Comparatively speaking, the condition of changing critical path is only calculated once when the objective function is given based on ICP method, it is also obvious that the ICP method calculation is simpler than DBICP method.

V. CONCLUSION

Considering the risk factors, the interval numbers is used to denote uncertain activity time of a project. Compared with the probability method, interval number needs less priori data to assume the probability distribution, and it can be easily acquired by personal judgment.

The network which considered the risk factors is called risk element network, and some definitions about it are given. Based on LP method, ICP method is proposed. The model can calculate the total interval duration time of the network with different ICPs acquired by different preference indexes. Meanwhile, by analyzing the sensitivity coefficient, the control priorities of sub-path also can be acquired.

From the example and the analysis, traditional probability PERT method can deal with the uncertain

problem of the network, but it may optimistically estimate the project duration time, namely, the duration time is underestimate. But ICP method could give more accurate duration time and critical path.

Otherwise, compared with DBICP method, ICP method could give more exact result and the calculation procedure is simpler.

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