

# Existence Theorem for Mean-Reverting CEV Process with Regime Switching

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**Abstract**—Empirical studies show that the most successful continuous-time models of the short term rate in capturing the dynamics are those that allow the volatility of interest changes to be highly sensitive to the level of the rate. The mean-reverting constant elasticity of variance (CEV) process with regime switching is a stochastic differential equation that has found considerable use as a model for interest rate, volatility, and other financial quantities. Since the coefficients of CEV process do not satisfy the linear growth condition, we can not examine its properties by traditional techniques. This paper overcomes the mathematical difficulties due to the nonlinear growth of the mean-reverting CEV process with regime switching, and provides a detailed proof that there is a unique positive global solution for such SDE.

**Keywords**—CEV process; global solution; Gronwall's inequality; Lipschitz condition; regime switching

## I. INTRODUCTION

Option pricing is one of the most important research fields in financial economics from both practical and theoretical point of view. The work of Black and Scholes [1] and Merton [2] laid the foundations of the research field and motivated important research in option pricing theory, its mathematical models and its computational techniques. The Black-Scholes-Merton formula is one of the important products of economic research of the last century and it has been widely adopted by traders, analysts, investors and other finance researcher.

Despite its popularity, the Black-Scholes-Merton formula is not without flaws. It has been documented in many studies in empirical finance that the Geometric Brownian Motion (GBM) assumed in the Black-Scholes-Merton model does not provide a realistic description for the behavior of asset price dynamics. One of substitutes is the CEV model, which is originally introduced by Cox [3] and Cox and Ross [4]. The main advantages of using the CEV model are that it can account for the implied volatility smile and smirk and that it can also capture the leverage effect associated with asset price. Many empirical studies have been conducted in the literature to justify the use of the CEV model, for instance, Mendoza-Arriaga and

Linetsky [5], Ruas, Dias and Nunes [6], Larguinho, Dias and Braumann [7], Thakoor, Tangman and Bhuruth [8].

Markovian regime-switching models have drawn a significant amount of attention in recent years due to their ability to include the influence of macroeconomic factors on individual asset price dynamics<sup>[9-13]</sup>. There are substantial empirical evidences in support of the existence of regime switching effects on stock market returns and default probabilities. Using the CRSP stock market returns over the period 1929-1989, Schaller and Norden [14] demonstrate that there is compelling evidence of regime switching in US stock market returns and the evidence for switching is robust to different specifications such as switching in means, switching in variances, and switching in both means and variances. Ang and Timmermann [15] show that regime-switching models can capture the stylized behavior of many asset returns such as fat tails, heteroskedasticity, and skewness.

In this paper, we investigate the global positive solution of a stochastic differential equation, where we generalize the mean-reverting CEV process by replacing the constant parameters with the corresponding parameters modulated by a continuous-time, finite-state, Markov chain. Since the coefficients of mean-reverting CEV process with regime switching do not satisfy the linear growth condition, so we can not examine its properties by traditional techniques. This paper overcomes the mathematical difficulties due to the nonlinear growth of the mean-reverting CEV process with regime switching, and provides a detailed proof that there is a unique positive global solution for such SDE.

This paper is organized as follows. In Section II, we develop a mean-reverting CEV process with regime switching. Since the proof of our main result is rather technical basic preliminaries and several lemmas are provided in Section III. In Section IV, we give our main result and show the detailed proof of the result. Conclusion is given in Section V.

## II. MEAN-REVERTING CEV PROCESSES WITH REGIME SWITCHING

Since the pioneer work of Hamilton [16], it has been

usually accepted that an unobserved Markov-switching process can be used to appropriately model the Gross Domestic Product (GDP). According to the GDP data (logarithms of postwar quarterly US real GDP from 1947.1 to 2009.4) that is downloaded from Federal Reserve Bank of St. Louis, although the output growth fluctuates around its sample mean, there are episodes of lower and even negative growth that coincide with NBER recessions as

show in Fig .1. Thus it is necessary to react the effects of regime switching in modelling asset price.

Throughout this paper, we let  $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)$  be a complete probability space with a filtration  $F_{t \geq 0}$

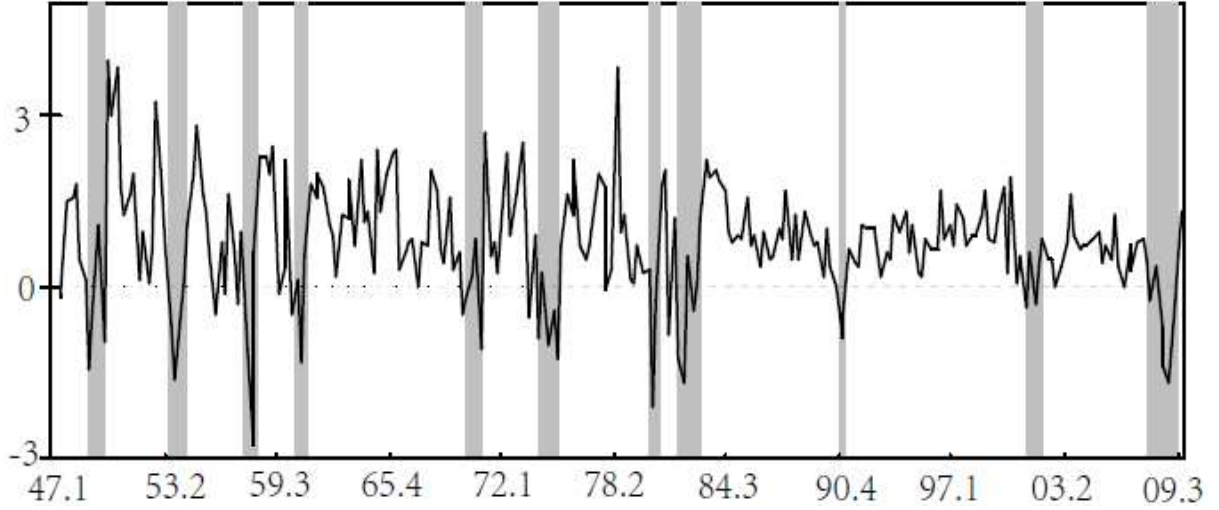


Figure 1. (growth rate) US Gross Domestic Product 1947.1-2009.4

satisfying the usual conditions (i.e., it is increasing and right continuous while  $F_0$  contains all  $P$ -null sets), upon which all stochastic processes are defined. Let  $X(t)$  be a finite-state continuous-time Markov chain taking values among  $G$  different states, where  $G$  is the total number of states considered in the economy. Each state represents a particular regime and is labeled by an integer  $i$  between 1 and  $G$ . Hence the state space of  $X(t)$  is given by  $M := \{1, 2, \dots, G\}$  which can be used to model factors of the economy such as GDP and stock price indices. One might interpret the states of  $X(t)$  as different stages of a business cycle. In economics, business cycles refer to the recurring and fluctuating levels of economical activities that an economic system undergoes over a long time period. For instance, there are usually five stages of a business cycle, namely, expansion, peak, recession, trough, and recovery. By interpreting the states of the Markov chain  $X(t)$  as different stages of a business cycle, one could suppose that  $G = 5$  and that state 1, state 2,  $L$ , and state 5 represent expansion, peak,  $L$ , and recovery, respectively.

To obtain the transition probabilities of the Markov chain  $X(t)$ , we need to specify its generator matrix  $Q$ . For easy exposition, we assume that a constant generator  $Q = (q_{ij})_{G \times G}$  is given. Clearly it is straightforward to extend the framework to the case of time varying generator. From Markov chain theory (see for example, Yin and Zhang [17]), the elements  $(q_{ij})_{G \times G}$  in the matrix  $Q$  satisfy:

- (1).  $q_{ij} \geq 0$  if  $i \neq j$ ;
- (2).  $q_{ii} \leq 0$  and  $q_{ii} = -\sum_{j \neq i} q_{ij}$  for each  $i = 1, L, G$ .

Assume that the Markov chain  $X(t)$  at any time  $t > 0$  is in a regime  $i \in M$ . Then after a period of time  $\Delta t$ , the Markov chain  $X_{t+\Delta t}$  may stay in regime  $i$  with probability  $P^X(i, i)$  or jump to any other regime  $j \in M$  with probability  $P^X(i, j)$ , where the one-step transition probabilities  $P^X(i, j)$  of the Markov chain  $X(t)$  are given by

$$p_{i,j}^X = P\{X_{t+\Delta t} = j | X_t = i\} = \begin{cases} e^{q_{ii}\Delta t}, & j = i, \\ (1 - e^{q_{ii}\Delta t}) \frac{q_{ij}}{-q_{ii}}, & j \neq i. \end{cases}$$

Let  $W(t)$  be a standard Brownian motion defined on the probability space  $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)$ . We consider the following regime-switching mean-reverting CEV process

$$dY(t) = a_{X(t)}(b_{X(t)} - Y(t))dt + \sigma_{X(t)}Y(t)^\beta dW(t), t \geq 0, \quad (1)$$

with initial values  $Y_0 = y_0$  and  $X_0 = x_0$ .  $Y(t)$  represents an underlying variable (for example, the stochastic interest rate or default intensity),  $a_{X(t)}$  denotes the speed of mean reversion,  $b_{X(t)}$  denotes the long term mean of the variable, and  $\sigma_{X(t)}$  is the volatility coefficient. The model parameters  $a_{X(t)}$ ,  $b_{X(t)}$  and  $\sigma_{X(t)}$  depend on the Markov chain  $X(t)$ , indicating that they can take different values in different regimes, where  $a_i$ ,  $b_i$  and  $\sigma_i$  are assumed to be positive for each  $i \in M$ .

### III. PRELIMINARIES AND SOME LEMMAS

Since the underlying variable  $Y(t)$  is mainly used to model stochastic volatility or interest rate or an asset price,

it is critical that  $Y(t)$  will never become negative. Mao et al. [17] discuss its analytical properties when  $\frac{1}{2} \leq \beta \leq 1$  and show that for given any initial data  $Y_0 = y_0 > 0$  and  $X_0 = x_0 \in M$ , the solution  $Y(t)$  of (1) will remain positive with probability 1, namely  $Y(t) > 0$  for all  $t \geq 0$  almost surely, if one of the following two conditions holds:

- (1).  $\frac{1}{2} < \beta \leq 1$ ;
- (2).  $\beta = \frac{1}{2}$  and  $\sigma_i^2 \leq 2a_i b_i$  for all  $i \in M$ .

When  $\beta > 1$ , the diffusion coefficient of (1) does not satisfy the linear growth condition, though it is locally Lipschitz continuous. We wonder if there exists a unique positive global solution and if the solution may explode at a finite time. Furthermore, since (1) is used to model interest rate and other quantities, it is critical that the solution  $Y(t)$  will never become negative. To prove there exists a unique positive global solution, we first establish the following lemmas that can be found in Mao et al. [18].

The integral inequalities of Gronwall-type have been widely applied in the theory of ordinary differential equations and stochastic differential equations to prove the results on existence, uniqueness, boundedness, comparison, continuous dependence, perturbation and stability etc. We establish the well-known inequalities of this type as follows.

**Lemma 1 (Gronwall's inequality)** Let  $T > 0$  and  $c > 0$ . Let  $u(\cdot)$  be a Borel measurable bounded non-negative function on  $[0, T]$ , and let  $v(\cdot)$  be a non-negative integrable function on  $[0, T]$ ,

$$u(t) \leq c + \int_0^t v(s)u(s)ds, \text{ for all } 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right), \text{ for all } 0 \leq t \leq T.$$

**Proof.** Without loss of generality we may assume that  $c > 0$ . Set

$$z(t) \leq c + \int_0^t v(s)u(s)ds, \text{ for all } 0 \leq t \leq T,$$

Then  $u(t) \leq z(t)$ . Moreover, by the chain rule of classical calculus, we have

$$\log(z(t)) = \log(c) + \int_0^t \frac{v(s)u(s)}{z(s)}ds \leq \log(c) + \int_0^t v(s)ds.$$

This implies

$$z(t) \leq c \exp\left(\int_0^t v(s)ds\right), \text{ for all } 0 \leq t \leq T.$$

Thus the required inequality follows since  $u(t) \leq z(t)$ .

**Lemma 2.** The coefficients of (1) satisfy the local Lipschitz condition for given initial value  $Y_0 = y_0 > 0$ , i.e., for every integer  $k > 1$ , there exists a positive constant  $L_k$  such that for all  $i \in M$ , and those  $x, y$  with  $x \in [0, k]$  and  $y \in [0, k]$ ,

$$|a_i(b_i - x) - a_i(b_i - y)| \leq L_k |x - y|, |\sigma_i x^\beta - \sigma_i y^\beta| \leq L_k |x - y|.$$

And thus there exists a unique local solution to (1).

#### IV. POSITIVE AND GLOBAL SOLUTION

**Theorem 2.** for any given initial value  $Y_0 = y_0 > 0$ ,  $a_i$ ,  $b_i$  and  $\sigma_i > 0$  for all  $i \in M$ , there exists a unique positive global solution  $Y(t)$  to (1) on  $t \geq 0$ .

**Proof.** According to (1), the local Lipschitz condition guarantees the existence of the unique local solution  $Y(t)$ ,  $t \in [0, \tau_e)$ , where  $\tau_e$  is the stopping time of the explosion or first zero time. To prove our theorem, we need to show that  $\tau_e = \infty$  a.s. If this is not true, then we can find a pair of positive constants  $\varepsilon$  and  $T$  such that

$$P(\tau_e \leq T) > \varepsilon.$$

For each integer  $k > 1$ , define the stopping time

$$\tau_k = \inf\{t \geq 0 \mid Y(t) \geq k\}.$$

Since  $\tau_k \rightarrow \tau_e$  almost surely, we can find a sufficiently large integer  $k_0$  for

$$P(\tau_k \leq T) > \frac{\varepsilon}{2}, \forall k \geq k_0.$$

For  $\theta_i$ ,  $\gamma_i > 0$ ,  $i \in M$ , let us define a function  $V \in C^2(R_+ \times M; R_+)$  as

$$V_i(Y) = \theta_i \sqrt{Y} + \gamma_i Y^{-2}, \quad (2)$$

which is continuously twice differentiable in  $Y$ . It is easy to see that  $V_i(Y) \rightarrow +\infty$  as  $Y \rightarrow +\infty$  or  $Y \rightarrow 0$ . For any  $0 < t < T$ ,  $i \in M$ , by the Itô formula,

$$dV_i(Y(t)) = LV_i(Y(t))dt + \sigma_i Y(t)^\beta \left( \frac{1}{2} \theta_i Y(t)^{-\frac{1}{2}} - 2\gamma_i Y(t)^{-3} \right) dW(t),$$

where  $LV_i(Y(t))$  is defined by

$$LV_i(Y(t)) = a_i(b_i - Y(t)) \left( \frac{1}{2} \theta_i Y(t)^{-\frac{1}{2}} - 2\gamma_i Y(t)^{-3} \right) + \frac{1}{2} \sigma_i^2 Y(t)^{2\beta} \left( -\frac{1}{4} \theta_i Y(t)^{-\frac{3}{2}} + 6\gamma_i Y(t)^{-4} \right) + \sum_{j=1}^G q_{ij} V_j(Y(t)).$$

By boundedness of polynomial, it is easy to see that there exists a constant  $K$  such that

$$a_i(b_i - Y(t)) \left( \frac{1}{2} \theta_i Y(t)^{-\frac{1}{2}} - 2\gamma_i Y(t)^{-3} \right) + \frac{1}{2} \sigma_i^2 Y(t)^{2\beta} \left( -\frac{1}{4} \theta_i Y(t)^{-\frac{3}{2}} + 6\gamma_i Y(t)^{-4} \right) \leq K,$$

and

$$\sum_{j=1}^G q_{ij} V_j(Y(t)) \leq KV_i(Y(t)).$$

Therefore, for any  $t \in [0, T]$ ,

$$EV_{X_{\{t \wedge \tau_k\}}}(Y_{\{t \wedge \tau_k\}}) = V_{x_0}(y_0) + E \int_0^{t \wedge \tau_k} LV_{X(s)}(Y(s))ds \leq V_{x_0}(y_0) + KT + KE \int_0^t EV_{X_{\{s \wedge \tau_k\}}}(Y_{\{s \wedge \tau_k\}})ds$$

The Gronwall inequality implies

$$EV_{X_{\{T \wedge \tau_k\}}}(Y_{\{T \wedge \tau_k\}}) \leq [V_{x_0}(y_0) + KT]e^{KT}.$$

So

$$E[I_{\{\tau_k \leq T\}} V_{X_{\tau_k}}(Y_{\tau_k})] \leq [V_{x_0}(y_0) + KT]e^{KT}.$$

On the other hand, if we define

$$M_k = \inf\{V_i(Y(t)) \mid Y(t) > k, t \in [0, T], i \in M\},$$

then  $M_k \rightarrow +\infty$ . It now follows from (3.40) and (3.41) that

$$[V_{x_0}(y_0) + KT]e^{KT} \geq M_k P\{\tau_k \leq T\} \geq \frac{1}{2} \varepsilon M_k.$$

Letting  $k \rightarrow +\infty$  yields a contradiction so we must have  $\tau_e = +\infty$  a.s.

Thus there exists a unique positive global solution  $Y(t)$  to (1) on  $t \geq 0$ .

## V. CONCLUSION

In this paper, we propose a mean-reverting CEV process that has switching dynamics governed by a continuous-time finite state Markov chain. We investigate the global positive solution of a stochastic differential equation, Since the coefficients of mean-reverting CEV process with regime switching do not satisfy the linear growth condition, This paper overcomes the mathematical difficulties due to the nonlinear growth of the mean-reverting CEV process with regime switching, and provides a detailed proof that there is a unique positive global solution for such SDE.

For further research it would be worth considering convergence of Monte Carlo simulations based on natural Euler-Maruyama discretization involving the mean-reverting CEV process with regime switching.

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## REFERENCES

- [1]. F. Black, M. Scholes. "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol 81,1973, pp. 637–659.
- [2]. R. Merton. "Theory of rational option pricing," *Bell Journal of Economics and Management Science*, vol 4,1973, pp. 141–183.
- [3]. J. Cox, "Notes on Option Pricing I: Constant Elasticity of Variance Diffusions," Stanford university, Working paper, 1975.
- [4]. J. Cox, S. Ross. "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, vol 3, 1976, pp. 145–166.
- [5]. Mendoza-Arriaga R, Linetsky V. "Pricing equity default swaps under the jump-to-default extended CEV model." *Finance and Stochastics*, vol 15, 2011, pp. 513-540.
- [6]. Ruas J P, Dias J C, Nunes J P V. "Pricing and static hedging of American-style options under the jump to default extended CEV model." *Journal of Banking & Finance*, vol 37, 2013, pp. 4059-4072.
- [7]. Larguinho M, Dias J C, Braumann C A. "On the computation of option prices and Greeks under the CEV model." *Quantitative Finance*, vol 13, 2013, pp. 907-917.
- [8]. Thakoor N, Tangman D Y, Bhuruth M. "A new fourth-order numerical scheme for option pricing under the CEV model." *Applied Mathematics Letters*, vol 26, 2013, pp. 160-164.
- [9]. Ang A., Bekaert. G. "Regime switches in interest rates." *Journal of Business and Economic Statistics*, vol 20, 2012, pp. 163-182.
- [10]. Boyle. P., Draviam, T. "Pricing exotic options under regime switching." *Insurance: Mathematics and Economics*, vol 40, 2007, pp. 267-282.
- [11]. Siu, T. K., Erlwein, C., Mamon, R. S. "The pricing of credit default swaps under a markov-modulated merton's structural model." *North American Actuarial Journal*, vol 12, 2008, pp. 18-46.
- [12]. Yao, D. D., Zhang, Q., Zhou, X. Y. A regime-switching model for European options. In Houmin Yan, Yin, G. G. and Zhang, Q. (Ed.), *Stochastic processes, optimization, and control theory: applications in financial engineering, queueing networks, and manufacturing systems*, 2006, pp. 281-300. Berlin: Springer.
- [13]. Maalaoui Chun, O., Dionne, G., Fran\_cois, P. "Credit spread changes within switching regimes." *Journal of Financial and Quantitative Analysis*, 2014, forthcoming.
- [14]. Schaller H, Norden SV. "Regime switching in stock market returns." *Applied Financial Economics*, vol 7, 1997, pp. 177-191.
- [15]. Ang A, Timmermann A. "Regime changes and financial markets." *Annual Review of Financial Economics*, vol 4, 2012, pp. 313-337.
- [16]. Hamilton, J. D. "A new approach to the economic analysis of nonstationary time series and the business cycle." *Econometrica*, vol 57, 1989, pp. 357-384.
- [17]. Yin G G, Zhang Q. *Continuous-time Markov chains and applications: a two-time-scale approach*. Springer Science & Business Media, 2012.
- [18]. Mao, Xuerong, Chenggui Yuan. *Stochastic differential equations with Markovian switching*. London: Imperial College Press, 2006.