Solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions

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Abstract. In this paper, we investigate a coupled system with integral boundary conditions of nonlinear fractional differential equations. Using Leggett-Williams theorem, the existence of solutions is obtained. An example is given to illustrate our result.

Introduction

Fractional calculus has recently evolved as an interesting field of research. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. For more details and examples, see [1-10] and the references therein. However, as far as we know, few results can be found in the literature concerning a coupled system of nonlinear fractional differential equations with integral boundary value problems. As a result, this paper is concerned with the existence of solutions for the following coupled system with integral boundaryconditions

$$\begin{aligned} &D_{0+}^{\alpha_1}u(t)=f_1(t,u(t),v(t),K_1u(t),H_1v(t)),\ \ 0<t<1,\\ &D_{0+}^{\alpha_2}v(t)=f_2(t,u(t),v(t),K_2u(t),H_2v(t)),\ \ 0<t<1,\\ &u(0)=u'(0)=0,\quad u(1)=\lambda_1\int_0^1 u(s)\mathrm{d}s,\\ &v(0)=v'(0)=0,\quad v(1)=\lambda_2\int_0^1 v(s)\mathrm{d}s. \end{aligned} \tag{1}$$

Where $2<\alpha_i\le 3,\ 0<\lambda_i<\alpha_i,\ D_{0+}^{\alpha_1},\ D_{0+}^{\alpha_2}$ are the Caputo fractional derivative, $k_i(t,s):[0,1]\times[0,t]\to R,\quad h_i(t,s):[0,1]\times[0,t]\to R$,

$$(K_ix)(t)=\int_0^t k_i(t,s)x(s)\mathrm{d}s,\ \ (H_ix)(t)=\int_0^t h_i(t,s)x(s)\mathrm{d}s,$$

be continuous functions, and

$$l_i=\sup_{t\in[0,1]}\int_0^t k_i(t,s)\mathrm{d}s<\infty,\quad m_i=\sup_{t\in[0,1]}\int_0^t h_i(t,s)\mathrm{d}s<\infty,\ \ i=1,2.$$

Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper. These definitions can be found in works [11, 12].

Definition 2.1 The fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s,$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 For a continuous function $y:(0,\infty)\to R$, the Caputo derivative of fractional Order $\alpha>0$ of is defined as

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{(n)}\int_0^t(t-s)^{n-\alpha-1}y(s)\mathrm{d}s,$$

Where $n=[\alpha]+1$, provided the right side is pointwise defined on $(0,\infty)$.

$(u(t),v(t))$ is a solution of the coupled system (1) if and only if $(u(t),v(t))$ is a solution of the system of integral equations

$$u(t)=\int_0^1 G_1(t,s)\, f_1(s,u(s),v(s),K_1u(s),H_1v(s))\mathrm{d}s,$$

$$v(t)=\int_0^1 G_2(t,s)\, f_2(s,u(s),v(s),K_2u(s),H_2v(s))\mathrm{d}s,$$

where $G_i(t,s)$ $(i=1,2)$ is the Green's function [1] defined as follows:

$$G_i(t,s)=\begin{cases}\dfrac{t^{\alpha_i-1}(1-s)^{\alpha_i-1}(\alpha_i-\lambda_i+\lambda_i s)-(\alpha_i-\lambda_i)(t-s)^{\alpha_i-1}}{(\alpha_i-\lambda_i)\Gamma(\alpha_i)}, 0\le s\le t\le 1,\\ \dfrac{t^{\alpha_i-1}(1-s)^{\alpha_i-1}(\alpha_i-\lambda_i+\lambda_i s)}{(\alpha_i-\lambda_i)\Gamma(\alpha_i)}, 0\le t\le s\le 1.\end{cases}$$

Lemma 2.1 Fix $2<\alpha_i\le 3,\ 0<\lambda_i<\alpha_i$, the Green's function $G_i(t,s)\ge 0$, for all $t,s\in[0,1]$, and satisfied the following inequalities: $t^{\alpha_i-1}G_i(1,s)\le G_i(t,s)\le \dfrac{\alpha_i}{\lambda_i}G_i(1,s), \forall t,s\in(0,1)$, where

$$G_i(1,s)=\frac{\lambda_i s(1-s)^{\alpha_i-1}}{(\alpha_i-\lambda_i)\Gamma(\alpha_i)},\ i=1,2.$$

Let E be a real Banach space and $P\subset E$ be a cone. The map θ is said to be a nonnegative continuous concave functional on P provided that $\theta: P\to[0,+\infty)$ is continuous and

$$\theta(tx+(1-t)y)\ge t\theta(x)+(1-t)\theta(y)$$

for all $x,y\in P$ and $0\le t\le 1$.

Let $0<a<b$ be given and let θ be a nonnegative continuous concave functional on P. Define the convex sets P_r and $P(\theta,a,b)$ by

$$P_r=\{x\in P,\ \|x\|\le r\},\ P(\theta,a,b)=\{x\in P\mid a\le\theta(x),\ \|x\|\le b\}.$$

Lemma 2.2 (Leggett-Williams fixed point theorem [13]) Let P be a cone in a real Banach space E, $T:\overline{P}_c\to\overline{P}_c$ be a completely continuous operator and let θ be a nonnegative continuous concave functional on P such that $\theta(x)\le\|x\|$ for all $x\in\overline{P}_c$. Suppose there exist $0<a<b<d\le c$ such that

(C1) $\{x\in P(\theta,b,d)\,|\,\theta(x)>b\}\ne\varnothing$ and $\theta(Tx)>b$ for $x\in P(\theta,b,d)$,

(C2) $\|Tx\|<a$ for $\|x\|\le a$,

(C3) $\theta(Tx)>b$ for $x\in P(\theta,b,c)$ with $\|Tx\|>d$,

Then T has at least three fixed point x_1,x_2 and x_3 such that

$$\|x_1\|<a,\ b<\theta(x_2)\ \text{and}\ \|x_3\|>a\ \text{with}\ \theta(x_3)<b.$$

If $b=c$, then condition (C1) implied condition (C3).

Main results

Theorem 3.1 Suppose $f_i:[0,1]\times[0,\infty)\times R^3\to[0,\infty)$ are continuous, there exist

$$0<a<b<\frac{b}{\min\{\varepsilon_1^{\alpha_1-1},\varepsilon_1^{\alpha_2-1}\}}=d\le c$$

such that

(A1) $f_i\le\dfrac{a}{\eta_i}$, for any $t\in[0,1],\ 0\le u(t)\le a,\ 0\le v(t)\le a,$

(A2) $f_i>\dfrac{d\alpha_i}{\lambda_i\eta_i}$, for any $t\in[\varepsilon_1,1-\varepsilon_2],\ b\le u(t)\le d,\ b\le v(t)\le d$, where $\varepsilon_1>0,\ \varepsilon_2>0,\ \varepsilon_1+\varepsilon_2<1.$

(A3) $f_i\le\dfrac{c}{\eta_i}$, for any $t\in[0,1],\ 0\le u(t)\le c,\ 0\le v(t)\le c$, where $\eta_i=\dfrac{\alpha_i}{\lambda_i}\displaystyle\int_0^1 G_i(1,s)\mathrm{d}s,$

then the coupled system (1) have at least three solutions $(u_1,v_1),\ (u_2,v_2),\ (u_3,v_3)$ such that

$$\|(u_1,v_1)\|<a,\ b<\theta(u_2,v_2) \text{ and } \|(u_3,v_3)\|>a \text{ with } \theta(u_3,v_3)<b.$$

Proof. Let $X=C[0,1]$ endowed with the norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$. Define $E=X\times X$ endowed with the norm $\|(u,v)\|_E=0.5(\|u\|+\|v\|)$, operator $T:E\to E$.

$$T(u,v)(t)=(T_1(u,v),T_2(u,v))$$

where

$$T_1(u,v)=\int_0^1 G_1(t,s)f_1(s,u(s),v(s),K_1u(s),H_1v(s))\mathrm{d}s,$$

$$T_2(u,v)=\int_0^1 G_2(t,s)f_2(s,u(s),v(s),K_2u(s),H_2v(s))\mathrm{d}s.$$

Then we define

$$P=\{(u,v)\,|\,(u,v)\in E,\ u(t)\ge 0,\ v(t)\ge 0,\ t\in[0,1]\}.$$

Obviously, P is a cone in the Banach space E.
Define

$$\theta(u,v)=0.5\left(\inf_{t\in[\varepsilon_1,1-\varepsilon_2]}u(t)+\inf_{t\in[\varepsilon_1,1-\varepsilon_2]}v(t)\right),\ \forall(u,v)\in P..$$

θ be a nonnegative continuous concave functional on P and

$$\theta(u,v)\le\|(u,v)\| \quad \text{for all } (u,v)\in\overline{P}_c.$$

Firstly, we will prove that $T:\overline{P}_c\to\overline{P}_c$

For any $(u,v)\in\overline{P}_c$, applying the condition (A3), we get

$$T_1(u(t),v(t))=\int_0^1 G_1(t,s)f_1\mathrm{d}s\le\frac{c}{\eta_1}\frac{\alpha_1}{\lambda_1}\int_0^1 G_1(1,s)\mathrm{d}s\le c.$$

Similarly, we have $|T_2(u(t),v(t))|\le c$, that is, $\|T(u,v)\|\le c$, therefore, $T:\overline{P}_c\to\overline{P}_c$. Using the same method, we can prove that (C2) in lemma 2.2 is satisfied. Applying the Arzela-Ascoli theorem and standard arguments, we conclude that T is a completely continuous operator.

Next, we will prove that (C1) in lemma 2.2 is satisfied. Let

$$(u_0,v_0)=(\frac{b+d}{2},\frac{b+d}{2}),$$

Then $(u_0,v_0)\in P(\theta,b,d)$ and $\theta(u_0,v_0)>b$. For any $(u,v)\in P(\theta,b,d)$, applying the condition (A2) and lemma 2.1, for $\forall t\in[\varepsilon_1,1-\varepsilon_2]$, we have

$$T_1(u(t),v(t))=\int_0^1 G_1(t,s)f_1\mathrm{d}s>\frac{d\alpha_1t^{\alpha_1-1}}{\lambda_1\eta_1}\int_0^1 G_1(1,s)\mathrm{d}s\ge\varepsilon_1^{\alpha_1-1}d\ge b,$$

hence we get $\inf_{t\in[\varepsilon_1,1-\varepsilon_2]}T_1(u(t),v(t))>b.$ Similarly, $\inf_{t\in[\varepsilon_1,1-\varepsilon_2]}T_2(u(t),v(t))>b.$ Thus, $\theta(T(u,v))>b.$ So (C1) in lemma 2.2 is satisfied.

Finally, we prove that (C3) in lemma 2.2 is satisfied.

For any $(u,v)\in P(\theta,b,c)$ and $\|T(u,v)\|_E>d$ applying lemma 2.1, for $\forall t\in[\varepsilon_1,1-\varepsilon_2]$, we have

$$\begin{aligned}T_1(u(t),v(t))&=\int_0^1 G_1(t,s)f_1\mathrm{d}s\ge\varepsilon_1^{\alpha_1-1}\int_0^1 g_1(s)f_1\mathrm{d}s\\&\ge\varepsilon_1^{\alpha_1-1}\sup_{t\in[0,1]}\int_0^1 G_1(t,s)f_1\mathrm{d}s=\varepsilon_1^{\alpha_1-1}\|T_1(u(t),v(t))\|,\end{aligned}$$

Then we get

$$\inf_{t\in[\varepsilon_1,1-\varepsilon_2]}T_1(u(t),v(t))\ge\varepsilon_1^{\alpha_1-1}\|T_1(u(t),v(t))\|.$$

Similarly,

$$\inf_{t\in[\varepsilon_1,1-\varepsilon_2]}T_2(u(t),v(t))\ge\varepsilon_1^{\alpha_2-1}\|T_2(u(t),v(t))\|.$$

It is easy to see that

$$
\begin{aligned}
\theta(T(u,v)) &= 0.5\{\inf_{t\in[\varepsilon_1,1-\varepsilon_2]} T_1(u(t),v(t)) + \inf_{t\in[\varepsilon_1,1-\varepsilon_2]} T_2(u(t),v(t))\} \\
&\geq 0.5\{\ \varepsilon_1^{\alpha_1-1}\| T_1(u(t),v(t))\| + \varepsilon_1^{\alpha_2-1}\| T_2(u(t),v(t))\|\} \\
&\geq \min\{\varepsilon_1^{\alpha_1-1},\varepsilon_1^{\alpha_2-1}\}\| T(u,v)\|_E > b.
\end{aligned}
$$

Thus (C3) in lemma 2.2 is satisfied.

Consequently, The coupled system (1) have at least three solutions (u_1,v_1), (u_2,v_2) and (u_3,v_3) such that

$$\|(u_1,v_1)\| < a,\ b < \theta(u_2,v_2) \ \text{ and } \ \|(u_3,v_3)\| > a \ \text{ with } \ \theta(u_3,v_3) < b.$$

Example

Let us consider the following coupled system

$$
\begin{aligned}
&D_{0+}^{2.5}u(t) = f_1(t,u(t),v(t),K_1u(t),H_1v(t)), \quad 0<t<1, \\
&D_{0+}^{2.6}v(t) = f_2(t,u(t),v(t),K_2u(t),H_2v(t)), \quad 0<t<1, \\
&u(0)=u'(0)=0, \quad u(1)=1.5\int_0^1 u(s)\mathrm{d}s, \\
&v(0)=v'(0)=0, \quad v(1)=1.6\int_0^1 v(s)\mathrm{d}s.
\end{aligned}
$$

where

$$
\begin{aligned}
&K_1u(t)=\int_0^t \sin(t-s)u(s)\mathrm{d}s, \quad K_2u(t)=\int_0^t \cos(t-s)u(s)\mathrm{d}s, \\
&H_1v(t)=\int_0^t e^{(t-s)}v(s)\mathrm{d}s, \quad H_2v(t)=\int_0^t e^{0.1(t-s)}v(s)\mathrm{d}s,
\end{aligned}
$$

$$
f_1=\begin{cases}
\dfrac{u(t)}{\eta_1}, \quad t\in[0,1],\ 0\le u\le 0.2,\ 0\le v\le 0.2, \\
\dfrac{0.2}{\eta_1}+50\dfrac{(v(t)-0.2)}{\eta_1}, \quad t\in[0,1],\ 0.2\le u\le 1,\ 0.2\le v\le 1, \\
\dfrac{40.2}{\eta_1}+\dfrac{K_1u(s)}{2e\eta_1}+\dfrac{H_1v(s)}{2e\eta_1}+\dfrac{\cos t}{2e\eta_1}-\dfrac{e^t}{2e\eta_1}, \quad t\in[0,1],\ 1\le u\le 81,\ 1\le v\le 81,
\end{cases}
$$

$$
f_2=\begin{cases}
\dfrac{u(t)}{\eta_2}, \quad t\in[0,1],\ 0\le u\le 0.2,\ 0\le v\le 0.2, \\
\dfrac{0.2}{\eta_2}+50\dfrac{(v(t)-0.2)}{\eta_2}, \quad t\in[0,1],\ 0.2\le u\le 1,\ 0.2\le v\le 1, \\
\dfrac{\sin t}{4\eta_2}-\dfrac{e^{0.1t}-1}{4e^{0.1}\eta_2}, \quad t\in[0,1],\ 1\le u\le 81,\ 1\le v\le 81,
\end{cases}
$$

η_1, η_2 is defined in theorem 3.1, let $a=0.2,\ b=0.4^{1.6},\ d=1,\ c=81,\ [\varepsilon_1,1-\varepsilon_2]=[0.4,0.8]$, using theorem 3.1, we know that the above coupled system has at least three solutions.

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