On the Two-Equal-Disjoint Path Cover Problem of Crossed Cubes

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Abstract

Embedding of paths have attracted much attention in the parallel processing. Many-to-many communication is one of the most central issues in various interconnection networks. A graph G is globally two-equal-disjoint path coverable if for any two distinct pairs of vertices (u, v) and (w, x) of G, there exist two disjoint paths P and Q satisfied that (1) P joins u to v and Q joins w to x, (2) $|P| = |Q|$, and (3) $V(P \cup Q) = V(G)$. In this paper, we prove that CQ_n is globally 2-equal-disjoint path coverable for $n \geq 5$.

Keywords: Interconnection network; Crossed cube; disjoint path; k-equal-disjoint path cover, 2-equal-disjoint path coverable.

1. Introduction

For the graph definition and notation we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. A *path* of length k from x to y is a finite sequence of distinct vertices $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, where $x = v_0$, $y = v_k$, and $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq k$. For convenience, we use the sequence $\langle v_0, \ldots, v_i, P, v_j, \ldots, v_k\rangle$, where $P = \langle v_i, v_{i+1}, \ldots, v_j\rangle$ to denote the path $\langle v_0, v_1, v_2, \ldots, v_k\rangle$. Note that it is possible that the path P has length 0. We can also write the path $\langle v_0, v_1, v_2, \cdots, v_k\rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \cdots, v_j, P_2, v_t, \cdots, v_k\rangle$, where P_1 is the path $\langle v_0, v_1, \cdots, v_i\rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \cdots, v_t\rangle$. We use $d(u, v)$ to denote the distance between u and v, i.e., the length of the shortest path joining u and v.

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A path is a *Hamiltonian path* if it contains all vertices of G. A graph G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two distinct vertices. A *cycle* is a path (except the first vertex is the same as the last vertex) containing at least three vertices. A cycle of G is a *Hamiltonian cycle* if it contains all vertices. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Finding node-disjoint paths is one of the important issues of routing among nodes in various interconnection networks. Node-disjoint (abbreviated as disjoint) paths can be used to avoid communication congestion and provide parallel paths for an efficient data routing among nodes. Moreover, multiple disjoint paths can be more fault-tolerant of node or link failures and greatly enhance the transmission reliability. Disjoint paths generally fall into three categories: one-to-one, one-to-many, and many-to-many. The one-to-one disjoint path is built with one source and one destination. The one-to-many disjoint paths like a tree structure, they contain one source and many distinct destination nodes. The many-to-many disjoint paths involve $k, k \geq 1$, disjoint paths with k pairs distinct source and destination nodes.

A *disjoint path cover* in a graph G is to find disjoint paths containing all the vertices in G. For an embedding of linear arrays in a network, the cover implies every node can be participated in a pipeline computation. One-to-one disjoint path covers in recursive circulants [10] and one-to-many disjoint path covers in some hypercube-like interconnection networks [11] were studied. The many-to-many k-disjoint path cover is proposed by Park etc. in [12]. In this paper, we call such many-to-many k-disjoint path cover (abbreviated as k-disjoint path cover) as many-to-many k-equal-disjoint path cover (abbreviated as k-equal-disjoint path cover) that k disjoint paths have same lengths. The k disjoint paths with equal length implies that the parallel processing of k pipeline is guaranteed accurately. Furthermore, a graph is called globally k-equal-disjoint path coverable if there exists a k-equal-disjoint path cover for any k distinct source-destination pairs.

An n-dimensional crossed cube, CQ_n[3], is a variation of hypercube, which is derived from hypercube by changing the connection of some hypercube links. Though some topological properties of crossed cubes have been studied in the literature [2, 3, 4, 5, 6, 7, 8, 9, 15, 16]. In this paper, we prove that the crossed cube is globally two-equal-disjoint path coverable. In next section, we give the definition of two-equal-disjoint path coverable problem and Crossed Cubes. Then we prove that the crossed cube is globally two-equal-disjoint path coverable in the section 3. In the final section, we give the conclusion.

2. Preliminary

In this section, we will first give the definition of globally two-equal-disjoint path coverable problem of a graph G, and then we will give the relevant definitions in graph theory and the definition of the Crossed cubes.

Definition 1 *A graph G is (u, v, w, x)-two-equal-disjoint path coverable if there are two disjoint paths P and Q such that P joins the vertices u to v, Q joins the vertices w to x, and $V(P \cup Q) = V(G)$.*

Definition 2 *A graph G is globally two-equal-disjoint path coverable if for any two distinct pairs of vertices (u, v) and (w, x), the (u, v, w, x)-two-equal-disjoint path cover exist.*

To define the Crossed cubes, as proposed by Efe [3], the notion so called "pair related" relation is introduced.

Definition 3
Let $R = \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. Two two-digit binary strings $u = u_1u_0$ and $v = v_1v_0$ are pair related if and only if $(u, v) \in R$.

The following is the recursive definition of the n-dimensional Crossed cube CQ_n.

Definition 4 [3] *The Crossed cube CQ_1 is a complete graph with two nodes labelled by 0 and 1, respectively. For $n \geq 2$, an n-dimensional Crossed cube CQ_n consists of two $(n-1)$-dimensional sub-Crossed cubes, CQ^0_{n-1} and CQ^1_{n-1}, and a perfect matching between the nodes of CQ^0_{n-1} and CQ^1_{n-1} according to the following rule:*

Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}...u_0 : u_i = 0$ or $1\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}...v_0 : v_i = 0$ or $1\}$. The node $u = 0u_{n-2}u_{n-3}...u_0 \in V(CQ^0_{n-1})$ and the node $v = 1v_{n-2}v_{n-3}...v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if

(1) $u_{n-2} = v_{n-2}$ *if n is even, and*

(2) $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, *for* $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

If u and v are two adjacent vertices in CQ_n and j is the leftmost differing bit, we say that v is the j-neighbor of u. Moreover, $\overline{u}$ represents the $(n-1)$-neighbor of u in CQ_n. We then introduce a important fault Hamiltonian result for proving the main theorem in the next section of this paper. A graph G is k-fault Hamiltonian connected if for any faulty set $F \subset V(G) \cup E(G)$ such that $|F| \leq k$, $G - F$ is still Hamiltonian connected.

Lemma 1 [7] *CQ_n is $n-3$ fault Hamiltonian connected.*

3. Crossed cube is globally two-disjoint equal path coverable

As a starting point we present the lemma below which establishes the base case of Theorem 1.

Lemma 2 *CQ_3 and CQ_4 are not globally two-equal-disjoint path coverable.*

Proof. To prove this lemma, we give a counter example for each case. Given two pair of vertices $0, 1$ and $2, 3$, there is no two-equal-disjoint path and cover all vertices in CQ_3. Given two pair of vertices $0, 3$ and $4, 7$, there is no two-equal-disjoint path and cover all vertices in CQ_4 also. □

Lemma 3 *CQ_5 is globally two-equal-disjoint path coverable.*

Proof. To prove this case is very tedious. With long and detail discussion, we have completed theoretical proof for CQ_5. Nevertheless, we do not present it in this paper for reducing complexity. However, we can also verify this small case directly using computer. □

Next we formally show the main result that CQ_n, $n \geq 5$, is globally two-disjoint equal path coverable.

Theorem 1 *Crossed cube, CQ_n, is globally two-equal-disjoint path coverable for $n \geq 5$.*

Proof. We prove this theorem by induction on n. The base case is CQ_5. With Lemma 3, the base case holds. By induction hypothesis, we can assume that CQ_n is globally two-equal-disjoint path coverable. Now, we need to show that CQ_{n+1} is also globally two-equal-disjoint path coverable. Let (a, b) and (c, d) be two distinct source-destination pairs of CQ_{n+1}. In the following, we establish two disjoint paths P, Q of length $2^n - 1$ with end vertices (a, b) and (c, d), respectively. By the relative positions of the four vertices, we divide the proof into four cases as follows.
Case 1: a, b, c and d are all in same CQ_n, say CQ^0_n, of CQ_{n+1}.

By induction, there are two disjoint paths (a, P_0, b) and (c, Q_0, d) in CQ_n^0, where $|P_0| = |Q_0| = 2^{n-1} - 1$. Let (w, x) and (y, z) be two edge on P_0 and Q_0, respectively, and let $P_0 = (a, P_0^1, w, x, P_0^2, b)$ and $Q_0 = (c, Q_0^1, y, z, Q_0^2, d)$. By induction again, we have two disjoint paths P_1 and Q_1 of length $2^{n-1} - 1$ with end vertices $\overline{w}, \overline{x}, \overline{y}$ and $\overline{z}$ in CQ_n^1. Let $P = (a, P_0^1, w, \overline{w}, P_1, \overline{x}, x, P_0^2, b)$ and $Q = (c, Q_0^1, y, \overline{y}, Q_1, \overline{z}, z, Q_0^2, d)$. Clearly, P and Q are two disjoint paths and $|P| = |Q| = 2^n - 1$. (See Fig. 1)

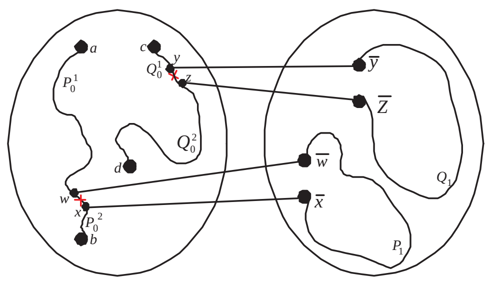

Figure 1: a, b, c and d are in CQ_n^0.

Case 2: a, b and c are in same CQ_n, say CQ_n^0, of CQ_{n+1}; d is in CQ_n^1.

Let x be a vertex in CQ_n^0 and $x \notin \{a, b, c, \overline{d}\}$. By hypothesis, there are two disjoint paths $< a, P_0, b >$ and $< c, Q_0, x >$ with $|P_0| = |Q_0| = 2^{n-1} - 1$ in CQ_n^0. Let w be the neighbor of b on the path P_0. If $\overline{w} \neq d$ and $\overline{b} \neq d$, we can get two disjoint paths $< \overline{w}, P_1, \overline{b} >$ and $< \overline{x}, Q_1, d >$ with $|P_1| = |Q_1| = 2^{n-1} - 1$ in CQ_n^1. (See Fig. 2) Let $P =< a, P_0, w, \overline{w}, P_1, \overline{b}, b >$ and $Q =< c, Q_0, x, \overline{x}, Q_1, d >$. In this case, P and Q are also two disjoint paths with $|P| = |Q| = 2^n - 1$. However, if $\overline{w} = d$ or $\overline{b} = d$. Choosing a and the neighbor of a on the path P_0 to replace b and w, we can rebuild another two-equal-disjoint path cover by the similar technique.

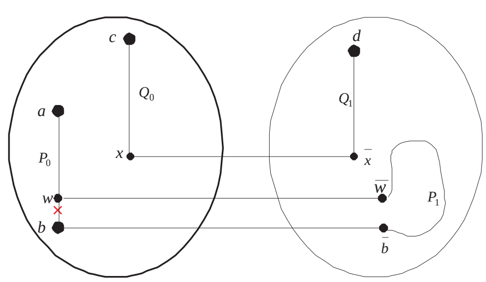

Figure 2: a, b and c are in CQ_n^0; d is in CQ_n^1.

Case 3: a and b are both in same CQ_n, say CQ_n^0, of CQ_{n+1}; c and d are both in CQ_n^1.

By Lemma 1, there exists a Hamiltonian path P (Q resp.) joining a (c resp.) and b (d resp.) in CQ_n^0 (CQ_n^1 resp.). (See Fig. 3)

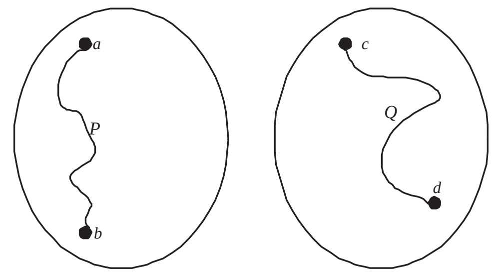

Figure 3: a and b are in CQ_n^0; c and d are in CQ_n^1.

Case 4: a and c are both in same CQ_n, say CQ_n^0, of CQ_{n+1}; b and d are both in CQ_n^1.

Let w and x be any two distinct vertices in CQ_n^0 except a and c and $\overline{w} \notin \{b, d\}$, $\overline{x} \notin \{b, d\}$. By hypothesis, there are two disjoint paths $< a, P_0, w >$ and $< c, Q_0, x >$ with $|P_0| = |Q_0| = 2^{n-1} - 1$ in CQ_n^0. Similarly, there are two disjoint paths $< \overline{w}, P_1, b >$ and $< \overline{x}, Q_1, d >$ with $|P_1| = |Q_1| = 2^{n-1} - 1$ in CQ_n^1. Let $P =< a, P_0, w, \overline{w}, P_1, b >$ and $Q =< c, Q_0, x, \overline{x}, Q_1, d >$. In this case, P and Q are also two disjoint paths with $|P| = |Q| = 2^n - 1$. (See Fig. 4)

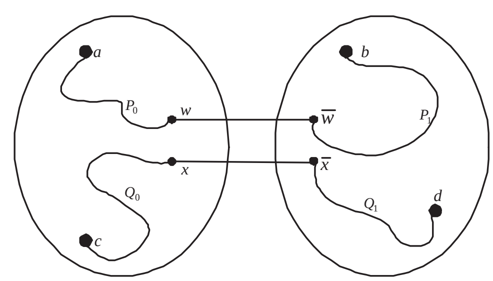

Figure 4: a and c are in CQ_n^0; b and d are in CQ_n^1.

□

4. Conclusion

In this paper, we discussed the two-equal-disjoint path coverable problem and proved that Crossed Cubes CQ_n are globally two-equal-disjoint path coverable for $n \geq 5$. The globally two-equal-disjoint path coverable problem is a extension of Hamiltonian connected problem. We can see Hamiltonian connected problem as globally one-path coverable problem, and then we extended this property to globally two-equal-disjoint path coverable. This work may help to discuss the many-to-many disjoint path coverable problem.

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