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Preliminary Behavior Analysis of Curvilinear Triangular Quadrature Elements

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Abstract. A weak form curvilinear triangular quadrature element is constructed and applied to the analysis of plane stress plates and torsional shafts. It is observed that convergence is achievable and the element is robust against domain irregularity, exhibiting both high accuracy and flexibility to complex geometry representation. Element-wise numerical differentiation and integration are carried out on the same set of points which is mapped from an octant of a sphere.

Introduction

With the increasing need for accurate simulation and great computational capacity, high-order methods have gained more and more attention in recent years, in both academia and industry. Consensus has formed that high-order methods are the way of the future. Traditional methods such as the p-version finite element [1], the spectral element [2], and the discontinuous Galerkin method [3] were boosted around 1980s and have been the driving forces for decades. Recent studies highlighted the spectral difference method and flux reconstruction schemes [4] which gather several related methods in a generic framework and still proliferate new efficient schemes. All these methods explicitly or implicitly use polynomials as the function space inside the element; as the degree of the polynomial p increases, the error decays exponentially. This represents the main attraction of high-order methods against the low-order counterparts in which only polynomial decay error is attainable.

The weak-form quadrature element method [5] is one such high-order method focusing on minimizing a scalar functional expressed in an integral form (usually for elliptic problems). As its name suggests, a high accuracy numerical quadrature rule stands in the center of the discretization scheme. Consequently all elemental degrees of freedom are pinned to a point set which is optimized for evaluation of the integral, thus obscuring the physical background which is already fully embedded in the functional integral. On one dimension (1D) segments and high dimensional tensor-product domains, the Legendre-Gauss-Lobatto (LGL) points accurately integrate the inner product of Lagrange interpolants acquired on the same set of points (collocated points), guaranteeing asymptotic stability (in practice, this scheme resembles the pseudo-spectral method which has become a standard C0 continuous element). Such a scheme has shown its versatility and efficiency in analysis of a variety of structures including beams [6], planar frames [7], plates [8], and even more superiorities in nonlinear analysis of geometrical exact beams and shells [9]. However, in simplicial domains, i.e. triangles, tetrahedra, etc., no analogues of LGL points have been found up to now. Search of good triangular point sets began decades ago and has progressed steadily in the intervening years (see [10-12] for reviews and recent results). Numerical point sets either of very low Lebesgue constant which bounds the interpolation error [11] or of nearly 100% integration efficiency [12] have been achieved, but the two criteria diverge unfortunately. To stabilize the element behavior at high orders, an over-sampled quadrature rule is often appealed to, which necessarily complicates the procedure and increases the computational cost. To the best knowledge of the authors, collocated point sets are only obtained at medium orders [13] due to the complex interference among boundary and interior interpolants. The element presented in this paper with collocated points is not stable at very high orders either [14], for the deficiency of integration strength limits the interpolation order below somewhere between 10 and 14. The present work sets out to extend the triangular quadrature element in [15] to curvilinear domains.

Construction of Point Set and Numerical Scheme

The detailed construction of the point set was described in [15]. Here it is briefly outlined that points on the triangle with area coordinates (ξ,η,ζ) satisfying $\xi+\eta+\zeta=1$ are in one-to-one correspondence with those on the first octant of a sphere with Cartesian coordinates (x,y,z) defined by $x^2+y^2+z^2=1$. Each point on the sphere surface divides the octant into three small spherical triangles intersected by great circle arcs; the spherical areas of the three triangles are i/n, j/n and k/n of that of the octant satisfying i+j+k=n, derived from an analogy with evenly spaced points on the flat triangle. On the edges, they degenerate to Chebyshev-Gauss-Lobatto points on a 1D segment. Although the points possess analytical coordinates, the interpolation and quadrature rule are not easily obtained. Two steps are employed to acquire differentiation matrices and quadrature rules. First, an orthonormal modal basis is evaluated and differentiated on the point set analytically. Second, the evaluation matrix (the so-called Vandermonde matrix) is numerically inverted to change the basis to the nodal interpolants.

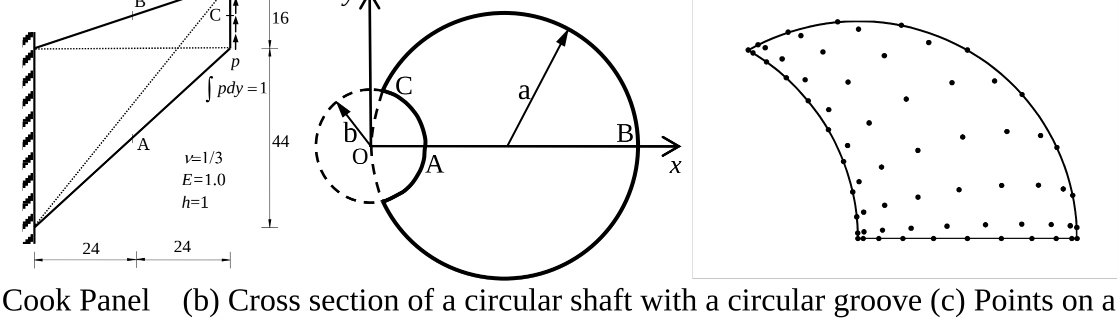
Numerical Results

The first example is a linear elastic plane stress problem. Cook panel is a benchmark problem to test the adaptability to irregular grids which low-order finite elements often lack. As shown in Fig. 1 (a), a quadrilateral elastic plate with the left edge clamped undergoes a parabolically distributed load on the right edge. Using a quadrilateral quadrature element, the irregularity is accounted for by a point-wise evolving geometrical transformation—Jacobian. In contrast, a triangle element adopts a constant Jacobian through the whole element as the transformation is linear. In two ways, S (along the short diagonal) and L (long diagonal), the quadrilateral panel is divided into two triangles. Computational results are given in Table 1.

Table 1. Cook panel divided into two triangular or one quadrilateral quadrature element

Order	v_C			$(\sigma_A)_{\max}$			$(\sigma_B)_{\min}$		
	Triangle S	Triangle L	Quad Ref.[17]	Triangle S	Triangle L	Quad Ref.[17]	Triangle S	Triangle L	Quad Ref.[17]
6	23.91	23.88	23.91	0.2383	0.2401	0.2337	-0.1959	-0.1959	-0.2018
8	23.86	23.88	23.93	0.2400	0.2599	0.2374	-0.2121	-0.2193	-0.2049
10	23.94	23.91	23.94	0.2457	0.2405	0.2364	-0.2031	-0.2069	-0.2028
12	23.99	23.93	23.95	0.2300	0.2433	0.2370	-0.2311	-0.2129	-0.2039
Ref.[16]		23.96			0.2362			-0.2023	

It is seen that the displacement at C in both mesh divisions S and L converges quickly as compared with that of the quadrilateral element. In contrast, the convergence rate of principal stresses at points A and B is relatively slow due to the loss of one-order of accuracy in the differentiation process, which, is found less dependent upon the irregularity of grids.



(a) Cook Panel (b) Cross section of a circular shaft with a circular groove (c) Points on a tenth order curved triangle (a=b=1)

Fig. 1. Numerical Examples

Curvilinear elements are commonplace in finite element analysis. In the second example, a prismatic shaft with curvilinear cross section (see Fig. 1(b)) under torsion is analyzed. The stress function ϕ satisfies a Poisson equation

$$\frac{\partial\phi}{\partial x^2}+\frac{\partial\phi}{\partial y^2}=-2, \qquad (1)$$

and vanishes on the two curved boundaries AC and CB. Taking advantage of the symmetry, the upper half is analyzed using a single triangular element. Discretized in the weak form, the functional involves only the inner product of the first-order derivative of ϕ, and only values of ϕ on AC and CB are specified.

Unlike a straight-edged element, the geometrical transformation of a curvilinear triangular element is not linear such that the Jacobian has to be evaluated at each quadrature point. Of crucial importance to convergence, this transformation needs to be as precise as possible, the best being conformal if existing, at least one-to-one and differentiable. A variety of such mappings are available, and for the sake of simplicity and generality, a blending function is chosen here

$$x=\zeta x_A+\xi x_B+\eta x_C+\frac{4\xi\eta}{1-(\eta-\xi)^2}\left[a+a\cos\left(\theta(1+\eta-\xi)\right)-\frac{1-(\eta-\xi)}{2}x_B-\frac{1+(\eta-\xi)}{2}x_C\right]$$
$$+\frac{4\zeta\eta}{1-(\eta-\zeta)^2}\left[b\cos\left(\theta\left(\frac{1+\eta-\zeta}{2}\right)\right)-\frac{1-(\eta-\zeta)}{2}x_A-\frac{1+(\eta-\zeta)}{2}x_C\right], \qquad (2)$$

where ξ,η and ζ are area coordinates corresponding to vertices B, C and A in the reference triangle, and $\theta=\cos^{-1}(b/2a)$ being the angle between OA and OC. Point set mapping is also dictated by the same blending function which distorts the distribution of points near the boundary severely. See Fig. 1(c). The distortion inevitably reduces the quadrature accuracy and may be more detrimental to the differentiation, since the Jacobian only re-weights the quadrature rule with the determinants but mixes derivatives of different directions.

Table 2. Computed Shear Stress at A on Curved Cross Section Prismatic shaft (*a*=1)

Order	*b/a*=0.3		*b/a*=1.0		*b/a*=1.7	
	τ_A	Error(%)	τ_A	Error(%)	τ_A	Error(%)
6	1.517	-10.78	1.053	5.34	0.318	6.09
8	1.745	2.67	1.026	2.57	0.303	1.11
10	1.655	-2.65	0.965	-3.49	0.342	14.04
12	1.672	-1.66	1.110	10.99	0.292	-2.80
Analytic	1.7		1.0		0.3	

To fully test the stability of the element, various ratios b/a and up to twentieth order point sets were used to compute the shear stress at point A which has a simple exact expression 2a-b [18]. In Table 2 results for typical b/a and medium orders are listed. At these medium orders, errors are generally on the percent level and therefore acceptable considering the inaccurate geometrical transformation. For even higher orders, results are similar without further convergence and hence omitted. Strong dependence of convergence rate on b/a is not observed, so grid quality is not the decisive factor. It is worth noticing that some unexpected large wrong numbers pop out randomly, indicating that large cancelations are lurking behind; amplified by the degenerate Jacobian at certain points, the differentiation becomes more unstable.

Conclusion

With a high-order approximation and the weak formulation ensuring conformity between elements, the curvilinear triangular quadrature elements are found to be immune to irregular meshing and mild distortion, as long as the geometric transformation is exact. In practice, inaccurate transformations are common and consequently may impede the convergence. Future research should focus on high-order geometric descriptions of curvilinear triangular elements in addition to development of new high-order stabilized point sets.

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