# The Globally Bi-3*-Connected Property of the Brother Trees

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## Abstract

Assume that $n$ is any positive integer. The brother tree $BT(n)$ is an interesting family of 3-regular planar bipartite graphs recently proposed by Kao and Hsu. In any $BT(n)$, we prove that there exist three internally-disjoint spanning paths joining $x$ and $y$ whenever $x$ and $y$ belong to different partite sets. Furthermore, for any three nodes $x$,$y$, and $z$ of the same partite set, there exist three internally-disjoint spanning paths of $BT(n) - \{z\}$ joining $x$ and $y$.
*Keywords:* hamiltonian, connectivity, container.

## 1 Introduction

In this paper, for the graph definitions and notations we follow [2]. $G = (V, E)$ is a *graph* if $V$ is a finite set and $E$ is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that $V$ is the *node set* and $E$ is the *edge set of* $G$. For any node $x$ of $V$, $\deg_G(x)$ denotes its degree in $G$. A graph $G$ is *cubic* if $\deg_G(x) = 3$ for any node $x \in V(G)$. Let $d_G(x, y)$ denote the distance between two nodes $x$ and $y$ in a graph $G$, and $D(G)$ denote the diameter of $G$. A bipartite graph $G = (V_1 \cup V_2, E)$ with bipartition $V_1$ and $V_2$ is a graph $G = (V, E)$ such that $V$ is the disjoint union of $V_1$ and $V_2$, and every edge of $G$ joins a node in $V_1$ and a node in $V_2$. We will use white to refer to a node in $V_1$ and black to refer to a node in $V_2$. A *path* $P$ joining $v_0$ and $v_k$ is represented by $\langle v_0, v_1, v_2, \ldots, v_k\rangle$. The *internal nodes* of $P$, $I(P)$, is the set $\{v_i \mid 0 < i < k\}$. Paths $P_1, P_2, \ldots, P_r$ are *internally-disjoint spanning* of $G$, if $I(P_i) \cap I(P_j) = \emptyset$ when $i \neq j$, and $P_1 \cup P_2 \cup \ldots \cup P_r$ spans $G$.

The *connectivity* of $G$, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Let $G = (V, E)$ be a graph with connectivity $\kappa(G) = k$. A *k-container* $C(x, y)$ in a graph $G$ is a set of $k$ internal node-disjoint paths between $x$ and $y$. It follows from Menger's Theorem [6] that there exists a $k$-container between any pair of nodes in a $k$-connected graph. In this paper, we are interested in another type of container. A $k^*$*-container* $C(x, y)$ in a graph $G$ is a $k$-container such that every node of $G$ is on some path in $C(x, y)$. In [1], Albert, Aldred, Holton and Sheehan first studied those cubic 3-connected graphs and proved that there exists a $3^*$-container between any pair of nodes. Such graphs are called *globally* $3^*$*-connected graphs.*

Since every globally $3^*$-connected graph is cubic, it contains an even number of nodes. Assume that $G = (V_1 \cup V_2, E)$ is a cubic 3-connected bipartite graph with bipartition $V_1$ and $V_2$ such that $|V_1| \geq |V_2| \geq 2$. Let $x$ and $y$ be any two distinct nodes in $V_2$. Assume that there exists a $3^*$-container $C(x, y) = \{P_1, P_2, P_3\}$ in $G$. Suppose that there are $a_i$ nodes of $V_1$ in $P_i$ for $i = 1, 2, 3$. Obviously, there are $a_i + 1$ nodes of $V_2$ in $P_i$ for $i = 1, 2, 3$. Hence, there are $a_1 + a_2 + a_3$ nodes of $V_1$ incidence with $P_1 \cup P_2 \cup P_3$ and there are $(a_1 + 1) + (a_2 + 1) + (a_3 + 1) - 4 = a_1 + a_2 + a_3 - 1$ nodes of $V_2$ incidence with $P_1 \cup P_2 \cup P_3$. Therefore, any cubic 3-connected bipartite graph is not globally $3^*$-connected.

For this reason, we say that a cubic bipartite graph $G = (V_1 \cup V_2, E)$ is *globally bi-*$3^*$*-connected* if there exists a $3^*$-container between any pair of nodes of the different partite sets. Obviously, $|V_1| = |V_2|$ in any globally bi-$3^*$-connected graph with bipartition $V_1$ and $V_2$. Furthermore, a globally bi-$3^*$-connected graph is *hyper* if there exists a $3^*$-container $C(x, y)$ in $G - \{z\}$ for any three nodes $x$, $y$, and $z$ of the same partite set of $G$. The concept of globally bi-$3^*$-connected and hyper globally bi-$3^*$-connected was proposed by Kao et al. [5]. It is proved that $G - \{e\}$ is hamiltonian for any $e \in E(G)$ if $G$ is globally bi-$3^*$-connected. Moreover, $G - \{x, y\}$ is hamiltonian for any $x \in V_1$ and $y \in V_2$ if $G$ is hyper globally bi-$3^*$ -connected. Kao et al. also proposed a family of hyper globally bi-$3^*$-connected graphs in [4].

Assume that $n$ is any positive integer. The brother tree $BT(n)$ is an interesting family of 3-regular planar bipartite graphs recently proposed by Kao and Hsu [3]. The number of nodes in $BT(n)$ is $6 \cdot 2^n - 4$ and the diameter is $2n + 1$. In [3], it is proved that $BT(n)$ is hamiltonian, and remains hamiltonian if any edge is deleted. Moreover, $BT(n)$ remains hamiltonian when a pair of nodes (one from each partite set) is deleted. In this paper, we prove that any brother tree $BT(n)$ is not

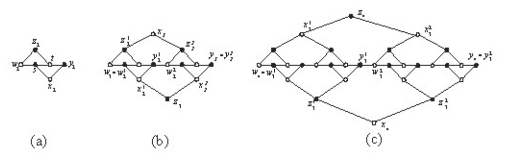

Figure 1: (a) BC(2), (b) BC(3), and (c) BC(4).

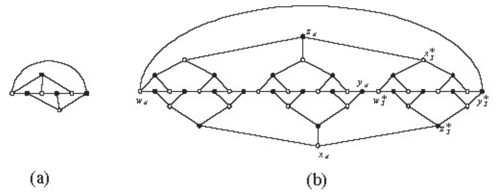

Figure 2: (a) BT(1), (b) BT(3).

only globally bi-3*-connected but also hyper globally bi-3*-connected. To our knowledge, $BT(n)$ is the only family of cubic planar bipartite graphs of the smallest diameter with such nice properties.

In the following section, we give the formal definition of brother tree and its properties. In section 3, we prove that any brother tree $BT(n)$ is globally bi-3*-connected. In section 4, we prove that any brother tree $BT(n)$ is hyper globally bi-3*-connected.

## 2 Brother trees

Assume that $k$ is an integer with $k \geq 2$. The *kth brother cell* $BC(k)$ is the five tuple $(G_k, w_k, x_k, y_k, z_k)$, where $G_k = (V, E)$ is a bipartite graph with bipartition $W$ (white) and $B$ (black) and $\{w_k, x_k, y_k, z_k\}$ is a set of four distinct nodes, called *corner nodes*. We can recursively define $BC(k)$ as follows:

[1] $BC(2)$ is the 5-tuple $(G_2, w_2, x_2, y_2, z_2)$ where $V(G_2) = \{w_2, x_2, y_2, z_2, s, t\}$, and $E(G_2) = \{(w_2, s), (s, x_2), (x_2, y_2), (y_2, t), (t, z_2), (w_2, z_2), (s, t)\}$.

[2] The $k$th brother cell $BC(k)$ with $k \geq 3$ is composed of two disjoint copies of $(k-1)$th brother cells $BC^1(k-1) = (G^1_{k-1}, w^1_{k-1}, x^1_{k-1}, y^1_{k-1}, z^1_{k-1})$ and $BC^2(k-1) = (G^2_{k-1}, w^2_{k-1}, x^2_{k-1}, y^2_{k-1}, z^2_{k-1})$, a white node $x_k$, and a black node $z_k$. To be specific, $V(G_k) = V(G^1_{k-1}) \cup V(G^2_{k-1}) \cup \{x_k, z_k\}$, $E(G_k) = E(G^1_{k-1}) \cup E(G^2_{k-1}) \cup \{(z_k, x^1_{k-1}), (z_k, x^2_{k-1}), (x_k, z^1_{k-1}), (x_k, z^2_{k-1}), (y^1_{k-1}, w^2_{k-1})\}$, $w_k = w^1_{k-1}$, and $y_k = y^2_{k-1}$.

$BC(2), BC(3)$, and $BC(4)$ are shown in Figure 1. We note that $BC^1(k-1)$ and $BC^2(k-1)$ are isomorphic for $k \geq 3$. This property is referred as the *symmetrical property* of $BC(k)$. For this reason, we define the degenerate case, $BC(1)$, as the 5-tuple $(G_1, w_1, x_1, y_1, z_1)$ as $V(G_1) = \{w_1, y_1\}$, $E(G_1) = \{(w_1, y_1)\}$ such that $x_1 = w_1$ and $y_1 = z_1$.

We can also define the brother cell $BC(k)$ from the *complete binary tree* $B(k)$, where $V(B(k)) = \{1, 2, \ldots, 2^k - 1\}$ and $E(B(k)) = \{(i, j) \mid \lfloor j/2 \rfloor = i\}$. Assume that $k$ is a positive integer with $k \geq 2$. The $k$th brother cell $BC(k) = (G_k, w_k, x_k, y_k, z_k)$ can be constructed by combining two $B(k)$'s, the *upper* tree $B(k)_u$ and the *lower* tree $B(k)_l$, and adding edges between their leaf nodes.

Let $n$ be a positive integer with $n \geq 1$. The *brother tree*, $BT(n)$, is composed of an $(n+1)$th brother cell $BC(n+1) = (G_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})$ and an $n$th brother cell $BC^*(n) = (G^*_n, w^*_n, x^*_n, y^*_n, z^*_n)$ with $V(G_{n+1}) \cap V(G^*_n) = \emptyset$. To be specific, $V(BT(n)) = V(G_{n+1}) \cup V(G^*_n)$ and $E(BT(n)) = E(G_{n+1}) \cup E(G^*_n) \cup \{(z_{n+1}, x^*_n), (y_{n+1}, w^*_n), (x_{n+1}, z^*_n), (w_{n+1}, y^*_n)\}$. $BT(1)$ and $BT(3)$ are shown in Figure 2. Obviously, $BT(n)$ is a 3-regular bipartite planar graph with $6 \cdot 2^n - 4$ nodes. Because the $(n+1)$th brother cell is composed of two disjoint $n$th brother cells and two terminals, the $n$th brother tree $BT(n)$ is composed of three disjoint $n$th brother cells, $BC^1(n)$, $BC^2(n)$, $BC^*(n)$ and two nodes, $\{x_{n+1}, z_{n+1}\}$. Moreover, $BC^1(n)$, $BC^2(n)$ and $BC^*(n)$ are arranged in a cyclic order in $BT(n)$. Thus any two nodes of $BT(n)$ are in the union of $BC^1(n)$, $BC^2(n)$, $BC^*(n)$ and $\{x_{n+1}, z_{n+1}\}$. For this reason, we can assume without loss of generality that any two nodes of $BT(n)$ are in $BC(n+1)$. This property is referred to as the *symmetrical property* of $BT(n)$.

The following lemmas are proved in [3].

**Lemma 1** *Assume that* $BC(n) = (G_n, w_n, x_n, y_n, z_n)$ *for some integer* $n \geq 2$.
*[1] There exists a hamiltonian path* $H$ *of* $BC(n)$ *joining any node in* $\{w_n, x_n\}$ *and any node in* $\{y_n, z_n\}$.
*[2] There exist two internally-disjoint spanning paths* $P$ *and* $Q$ *of* $BC(n)$ *such that* $P$ *joins* $w_n$ *and* $x_n$, *and* $Q$ *joins* $y_n$ *and* $z_n$.
*[3] There exist two internally-disjoint spanning paths* $R$ *and* $S$ *of* $BC(n)$ *such that* $R$ *joins* $w_n$ *and* $z_n$, *and* $S$ *joins* $x_n$ *and* $y_n$.

**Lemma 2** *Assume that* $n$ *is an integer with* $n \geq 2$. *Suppose that* $f$ *is any node of* $BC(n)$. *There exists a hamiltonian path* $H$ *of* $BC(n) - \{f\}$ *such that* $H$ *joins the two corner nodes of a different partite set containing c.*

## 3 The globally bi-3*-connected property of $BT(n)$

**Lemma 3** *Assume that* $n$ *is an integer with* $n \geq 2$. *Let* $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. *Suppose that* $c$ *is a node of* $BC(n)$. *Let* $p$ *and* $q$ *be the the corner nodes of the same partite set containing* $c$, *and* $r$ *be any corner* *node of a different partite set containing c. Then there exist three internally-disjoint spanning paths $P_1$, $P_2$, and $P_3$ such that (1) $P_1$ joins c and p, (2) $P_2$ joins c and q, and (3) $P_3$ joins c and r.*

**Lemma 4** *Assume that n is an integer with $n \geq 2$. Let $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. Suppose that c and d are two nodes from different partite set of $BC(n)$. At least one of the following cases holds:*
*[A] There exist four internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, and $P_4$ of $BC(n)$ such that (1) both $P_1$ and $P_2$ join c and d, (2) $P_3$ joins c and some corner node p of the same partite set as c, and (3) $P_4$ joins d and some corner node q of the same partite set as d.*
*[B] There exist five internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ of $BC(n)$ such that (1) $P_1$ joins c and d, (2) $P_2$ joins c and $w_n$, (3) $P_3$ joins d and $y_n$, and (4) $P_4$ joins c and p, and $P_5$ joins d and q where $\{p, q\} = \{x_n, z_n\}$.*
*[C] There exist five internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ of $BC(n)$ such that (1) both $P_1$ and $P_2$ join c and d, (2) $P_3$ joins c and p, and $P_4$ joins d and q where $p \in \{w_n, y_n\}$ and $q \in \{x_n, z_n\}$, and (3) $P_5$ joins the only node in $\{w_n, y_n\} - \{p\}$ and the only node in $\{x_n, z_n\} - \{q\}$.*

**Theorem 1** *$BT(n)$ is globally bi-$3^*$-connected for any positive integer n.*

# 4 The hyper globally bi-$3^*$-connected property of $BT(n)$

**Lemma 5** *Assume that n is an integer with $n \geq 2$. Let $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. Suppose that c and f are any two different nodes in the same partite set of $BC(n)$. Let p and q are corner nodes of the different partite set containing c. There exists three internally-disjoint spanning paths $P_1$, $P_2$, and $P_3$ of $BC(n) - \{f\}$ such that (1) $P_1$ joins c and p, (2) $P_2$ joins c and q, and (3) $P_3$ joins c and one of the corner node r of the same partite set containing c with $r \neq f$.*

**Lemma 6** *Assume that n is an integer with $n \geq 2$. Let $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. Suppose that c and d are any two different nodes in the same partite set of $BC(n)$. Then there exist four internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, and $P_4$ of $BC(n)$ such that (1) both $P_1$ and $P_2$ join c to d, and (2) $P_3$ joins c and p, and $P_4$ joins d and q where p and q are corner nodes in the same partite set containing c.*

**Lemma 7** *Assume that n is an integer with $n \geq 2$. Let $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. Suppose that $c, d$, and f are three nodes in the same partite set of $BC(n)$. Then at least one of the following cases holds in $BC(n) - \{f\}$:*
*[A] There exist four internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, and $P_4$ of $BC(n) - \{f\}$ such that (1) both $P_1$ and $P_2$ join c and d, and (2) $P_3$ joins c and p and $P_4$ joins d and q where $p \in \{w_n, x_n\}$ and $q \in \{y_n, z_n\}$.*
*[B] There exist five internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ of $BC(n) - \{f\}$ such that (1) $P_1$ joins c and d, (2) $P_2$ joins c and $w_n$, (3) $P_3$ joins d and $y_n$, and (4) $P_4$ joins c and p, and $P_5$ joins d and q where $\{p, q\} = \{x_n, z_n\}$.*
*[C] There exist five internally-disjoint spanning paths $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ of $BC(n) - \{f\}$ such that (1) both $P_1$ and $P_2$ join c and d, (2) $P_3$ joins c and p and $P_4$ joins d and q where $p \in \{w_n, y_n\}$ and $q \in \{x_n, z_n\}$, and (3) $P_5$ joins $\{w_n, y_n\} - \{p\}$ and $\{x_n, z_n\} - \{q\}$.*

**Theorem 2** *$BT(n)$ is hyper globally bi-$3^*$-connected for any positive integer n.*

**Proof.** We prove this theorem by induction. It is easy to check the theorem holds for $BT(1)$ and $BT(2)$. Assume that the theorem holds for $BT(n-1)$ with $n \geq 3$. By definition, $BT(n)$ is composed of an $(n+1)$th brother cell, denoted by $BC(n+1) = (G_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})$ and an $n$th brother cell, denoted by $BC^*(n) = (G_n^*, w_n^*, x_n^*, y_n^*, z_n^*)$. Let $c, d$, and $f$ be any three nodes of $BT(n)$ in the same partite set. We will show that $\{P_1, P_2, P_3\}$ forms a $3^*$-container between c and d in $BT(n) - \{f\}$. By the symmetrical property of $BT(n)$, we consider the following cases (1) $c, d$, and $f$ are in $BC(n+1)$, (2) $c, d$ are in $BC(n+1)$, and $f$ is in $BC^*(n)$.
Case 1: All $c, d$, and $f$ are in $BC(n+1)$. By Lemma 7, we have following three cases:
[A] There exist four internally-disjoint spanning paths $R_1$, $R_2$, $R_3$, and $R_4$ of $BC(n+1) - \{f\}$ such that (1) both $R_1$ and $R_2$ join c and d, and (2) $R_3$ joins c to p and $R_4$ joins d to q where $p \in \{w_{n+1}, x_{n+1}\}$ and $q \in \{y_{n+1}, z_{n+1}\}$. Suppose that $p = w_{n+1}$ and $q = y_{n+1}$. By Lemma 1, there exists a hamiltonian path S joining $y_n^*$ and $w_n^*$ of $BC^*(n)$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, w_{n+1}, y_n^*, S, w_n^*, y_{n+1}, R_4, d\rangle$. Suppose that $p = w_{n+1}$ and $q = z_{n+1}$. By Lemma 1, there exists a hamiltonian path S joining $y_n^*$ and $x_n^*$ of $BC^*(n)$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, w_{n+1}, y_n^*, S, x_n^*, z_{n+1}, R_4, d\rangle$. Suppose that $p = x_{n+1}$ and $q = y_{n+1}$. By Lemma 1, there exists a hamiltonian path S joining $z_n^*$ and $w_n^*$ of $BC^*(n)$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, x_{n+1}, z_n^*, S, w_n^*, y_{n+1}, R_4, d\rangle$. Suppose that $p = x_{n+1}$ and $q = z_{n+1}$. By Lemma 1, there exists a hamiltonian path S joining $z_n^*$ and $x_n^*$ of $BC^*(n)$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, x_{n+1}, z_n^*, S, x_n^*, z_{n+1}, R_4, d\rangle$.
[B] There exist five internally-disjoint spanning paths $R_1, R_2, R_3, R_4$, and $R_5$ of $BC(n+1) - \{f\}$ such that (1) $R_1$ joins c and d, (2) $R_2$ joins c and $w_{n+1}$, (3) $R_3$ joins d to $y_{n+1}$, and (4) $R_4$ joins c and p, and $R_5$ joins d and

$q$ where $\{p, q\} = \{x_{n+1}, z_{n+1}\}$. Suppose that $p = x_{n+1}$ and $q = z_{n+1}$. By Lemma 1, there exists two internally-disjoint spanning paths $S_1$ and $S_2$ of $BC^*(n)$ such that (1) $S_1$ joins $z_n^*$ and $w_n^*$ and (2) $S_2$ joins $y_n^*$ and $x_n^*$. We set $P_1 = R_1$, $P_2 = \langle c, R_2, w_{n+1}, y_n^*, S_2, x_n^*, z_{n+1}, R_5, d\rangle$, and $P_3 = \langle c, R_4, x_{n+1}, z_n^*, S_1, w_n^*, y_{n+1}, R_3, d\rangle$. Suppose that $p = z_{n+1}$ and $q = x_{n+1}$. By Lemma 1, there exists two internally-disjoint spanning paths $S_1$ and $S_2$ of $BC^*(n)$ such that (1) $S_1$ joins $x_n^*$ and $w_n^*$, and (2) $S_2$ joins $y_n^*$ and $z_n^*$. We set $P_1 = R_1$, $P_2 = \langle c, R_2, w_{n+1}, y_n^*, S_2, z_n^*, x_{n+1}, R_5, d\rangle$, and $P_3 = \langle c, R_4, z_{n+1}, x_n^*, S_1, w_n^*, y_{n+1}, R_3, d\rangle$.

[C] There exist five internally-disjoint spanning paths $R_1$, $R_2$, $R_3$, $R_4$, and $R_5$ of $BC(n+1) - \{f\}$ such that (1) both $R_1$ and $R_2$ join $c$ and $d$, (2) $R_3$ joins $c$ and $p$, and $R_4$ joins $d$ to $q$ where $p \in \{w_{n+1}, y_{n+1}\}$ and $q \in \{x_{n+1}, z_{n+1}\}$, and (3) $R_5$ joins $\{w_{n+1}, y_{n+1}\} - p$ and $\{x_{n+1}, z_{n+1}\} - q$. Suppose that $p = w_{n+1}$ and $q = x_{n+1}$. By Lemma 1, there exists two internally-disjoint spanning paths $S_1$ and $S_2$ of $BC^*(n)$ such that (1) $S_1$ joins $w_n^*$ and $z_n^*$, and (2) $S_2$ joins $y_n^*$ and $x_n^*$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, w_{n+1}, y_n^*, S_2, x_n^*, z_{n+1}, R_5, y_{n+1}, w_n^*, S_1, z_n^*, x_{n+1}, R_4, d\rangle$. Suppose that $p = w_{n+1}$ and $q = z_{n+1}$. By Lemma 1, there exists two internally-disjoint spanning paths $S_1$ and $S_2$ of $BC^*(n)$ such that (1) $S_1$ joins $w_n^*$ and $x_n^*$, and (2) $S_2$ joins $y_n^*$ and $z_n^*$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, w_{n+1}, y_n^*, S_2, z_n^*, x_{n+1}, R_5, y_{n+1}, w_n^*, S_1, x_n^*, z_{n+1}, R_4, d\rangle$. Suppose that $p = y_{n+1}$ and $q = x_{n+1}$. By Lemma 1, there exists two internally-disjoint spanning paths $S_1$ and $S_2$ of $BC^*(n)$ such that (1) $S_1$ joins $w_n^*$ and $x_n^*$, and (2) $S_2$ joins $y_n^*$ and $z_n^*$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, y_{n+1}, w_n^*, S_1, x_n^*, z_{n+1}, R_5, w_{n+1}, y_n^*, S_2, z_n^*, x_{n+1}, R_4, d\rangle$. Suppose that $p = y_{n+1}$ and $q = z_{n+1}$. By Lemma 1, there exists two internally-disjoint spanning paths $S_1$ and $S_2$ of $BC^*(n)$ such that (1) $S_1$ joins $w_n^*$ and $z_n^*$, and (2) $S_2$ joins $y_n^*$ and $x_n^*$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, y_{n+1}, w_n^*, S_1, z_n^*, x_{n+1}, R_5, w_{n+1}, y_n^*, S_2, x_n^*, z_{n+1}, R_4, d\rangle$.

Case 2: Both $c$ and $d$ are in $BC(n+1)$ and $f$ is in $BC^*(n)$. By Lemma 6, there exist four internally-disjoint spanning paths $R_1$, $R_2$, $R_3$, and $R_4$ of $BC(n+1)$ such that (1) both $R_1$ and $R_2$ join $c$ and $d$, (2) $R_3$ joins $c$ and $w_{n+1}$, and (3) $R_4$ joins $d$ and $x_{n+1}$. By Lemma 2, there exist a hamiltonian path $S$ joining $y_n^*$ and $z_n^*$ of $BC^*(n) - \{f\}$. We set $P_1 = R_1$, $P_2 = R_2$, and $P_3 = \langle c, R_3, w_{n+1}, y_n^*, S, z_n^*, x_{n+1}, R_4, d\rangle$.

The theorem is proved. □

# References


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