Uncertain Bonferroni Mean Operators

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Abstract

The Bonferroni mean is a traditional mean type aggregation operator bounded by the max and min operators, which is suitable to aggregate the crisp data. In this paper, we consider situations where the input data are interval numbers. We develop some uncertain Bonferroni mean operators, and then combine them with the well-known ordered weighted averaging operator and Choquet integral respectively for aggregating uncertain information. We also give their applications to multi-criteria decision making under uncertainty, and finally, some possible extensions for further research are discussed.

Keywords Bonferroni mean; interval number; multi-criteria decision making; ordered weighted averaging operator; Choquet integral.

1. Introduction

In many decision making situations under uncertainty, such as emergency management, military operations, contingency planning, and risk assessment, etc., interval numbers (interval utility values or value ranges) are usually used by experts to express their preference values over the considered objects, due to that the experts may not have enough knowledge or expertise about the problem domain or a decision should be made under time pressure and lack of data. In the process of decision making, how to aggregate or deal with these given uncertain data (interval numbers[1]) by using a proper aggregation operator or mathematical model becomes a key step. As a result, some basic operations like "addition", "subtraction", "multiplication" and "division" on interval numbers were introduced in Ref.2, which make a basis for the potential applications of uncertain data. The comparison and ranking of interval numbers is also a fundamental issue, which plays an important role in the aggregation of uncertain data. A variety of methods have been proposed to compare interval numbers.[3-5] Wang et al.[6] analyzed the strengths and weaknesses of the existing methods, and developed a simple yet practical preference ranking method of interval numbers. Facchinetti et al.,[7] and Xu and Da[8] also developed two straightforward possibility-degree formulas for the comparison between two interval numbers, and studied their desirable properties. Nevertheless, Xu and Chen[9] showed that the three possibility-degree formulae developed in Refs.6-8 are equivalent, and employed the uncertain weighted averaging operator[10] and the weights of experts to fuse all individual interval fuzzy preference relations into the collective interval fuzzy preference relation.

Based on the basic operations and the ranking methods of interval numbers, some useful uncertain aggregation operators have been put forward, including the uncertain ordered weighted averaging (UOWA) operator,[8] uncertain ordered weighted geometric (UOWG) operator,[11] continuous ordered weighted

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averaging (COWA) operator,[12] and continuous ordered weighted geometric (COWG) operator,[13] etc. Among them, the UOWA operator, which is motivated by the idea of the ordered weighted averaging (OWA) operator developed by Yager,[14] first ranks all the uncertain data in descending order, and then fuses all these ordered uncertain data together with the weights of their ordered positions. The UOWG operator utilizes the ordered weighted geometric (OWG) operator[15,16] and the possibility-degree formula to aggregate the given interval numbers. Yager[12] developed the COWA operator, which is an extension of the OWA operator to the case in which the given datum is an interval number rather than a finite set of data. The aggregated value derived by the COWA operator is associated with the attitudinal character of a basic unit-interval monotonic (BUM) function. The COWA operator is very suitable for aggregating decision information taking the form of interval fuzzy preference relation.[17] Based on the COWA operator and the geometric mean, Yager and Xu[13] introduced the COWG operator, and applied it to decision making with interval multiplicative preference relation. The common characteristic of the above uncertain aggregation operators are that they emphasize the importance of each datum or its ordered position, but can not reflect the interrelationships of the individual data.

The Bonferroni mean (BM), originally introduced by Bonferroni,[18] is a traditional mean type aggregation operator, which is suitable to aggregate the crisp data and can capture the expressed interrelationship between the individual data.[19] Recently, Yager[2] generalized the BM by replacing the simple average by other mean type operators such as the OWA operator[14] and Choquet integral[20] as well as associating different importance with the data. Considering the desirable property of the BM, and the need of extending its potential applications to more extensive areas, such as decision making under uncertainty, fuzzy clustering analysis, and uncertain programming, etc., in this paper, we extend the BM to aggregate uncertain data. In order to do so, we develop some uncertain BM operators, uncertain ordered weighted BM operator, and uncertain Bonferroni Choquet operator, etc., and study their properties. We also give their applications to multi-criteria decision making under uncertainty, and finally, discuss some possible extensions for further research.

2. A brief review on BMs

Given a collect of crisp data $a_i (i = 1, 2, \ldots, n)$, where $a_i \geq 0$, for all i, and $p, q \geq 0$. Bonferroni[18] originally introduced an aggregation operator, denoted as $B^{p,q}$ such that

$$B^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{1}$$

Recently, the operator $B^{p,q}$ has been discussed in Ref.19,21,22 and called Bonferroni mean (BM). For the special case where $p = q = 1$, the BM reduces to the following:[19]

$$B(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{n} \sum_{i=1}^{n} u_i a_i \right)^{\frac{1}{2}} \tag{2}$$

where $u_i = \dfrac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j$.

Yager[19] replaced the simple average used to obtain u_i by an OWA aggregation of all a_j $(j \neq i)$:

$$BON-OWA(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i OWA_\omega(v^i) \right)^{\frac{1}{2}} \tag{3}$$

where v^i is the $n-1$ tuple $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, ω is an OWA weighting vector of dimension $n-1$, with the components $\omega_k \geq 0$, $\sum_k \omega_k = 1$, and

$$OWA_\omega(v^i) = \sum_{k=1}^{n-1} \omega_k a_{\pi_i(k)} \tag{4}$$

where $a_{\pi_i(k)}$ is the kth largest element in the tupe v^i.

If each a_i has its personal importance, denoted by p_i, then (3) can be further generalized as:

$$BON-OWA_\omega(a_1, a_2, \ldots, a_n) = \left(\frac{1}{p} \sum_{i=1}^{n} p_i a_i OWA_\omega(v^i) \right)^{\frac{1}{2}} \tag{5}$$

where $p = \sum_{i=1}^{n} p_i$, $p_i \in [0,1]$, $i = 1, 2, \ldots, n$.

In multi-criteria decision making, let $\Omega = \{c_1, c_2, \ldots, c_m\}$ be a set of criteria, and let $\Omega^i = \Omega - \{c_i\}$ be the set of all criteria except c_i, then a monotonic set measure m_i over Ω^i is m_i : $2^{\Omega^i} \to [0,1]$, which has the properties: 1) $m_i(\phi)=0$, 2) $m_i(\Omega^i)=1$, and 3) $m_i(E) \ge m_i(F)$, if $F \subseteq E$. Using the measure m_i , Yager[19] further defined a Bonferroni Choquet operator as:

$$BON-CHOQ(a_1,a_2,...,a_n)=\left(\sum_{i=1}^{n}\frac{p_i}{p}a_iC_{m_i}(v^i)\right)^{\frac{1}{2}} \quad (6)$$

where

$$C_{m_i}(v^i)=\sum_{j=1}^{n-1}v_{ij}\left(m_i(H_j^i)-m_i(H_{j-1}^i)\right) \quad (7)$$

and H_j^i is the subset of Ω^i consisting the j criteria with the largest satisfactions, and $H_0^i=\phi$. $v_{i1},v_{i2},...,v_{in-1}$ are the elements in v^i , and these elements have been ordered so that $v_{ij1} \ge v_{ij2}$ if $j_1<j_2$.

In the next sections, we shall extend the above results to uncertain environments in which the input data are interval numbers.

3. UBM operators

An interval number can be defined as $\tilde{a}=[a^-,a^+]=\{t \mid 0 \le a^- \le t \le a^+\}$. Given two interval numbers $\tilde{a}_i=[a_i^-,a_i^+]\,(i=1,2)$, Xu and Zhai[2] gave the following operations:

1) $\tilde{a}_1+\tilde{a}_2=[a_1^-,a_1^+]+[a_2^-,a_2^+]=[a_1^-+a_2^-,a_1^++a_2^+]$.
2) $\tilde{a}_1\tilde{a}_2=[a_1^-,a_1^+]\cdot[a_2^-,a_2^+]=[a_1^-a_2^-,a_1^+a_2^+]$.
3) $\lambda\tilde{a}_1=\lambda[a_1^-,a_1^+]=[\lambda a_1^-,\lambda a_1^+]$, where $\lambda \ge 0$.
4) $\tilde{a}_1^{\lambda}=[a_1^-,a_1^+]^{\lambda}=[(a_1^-)^{\lambda},(a_1^+)^{\lambda}]$, where $\lambda \ge 0$.

To compare $\tilde{a}_i=[a_i^-,a_i^+]\,(i=1,2)$, we first calculate their expected values:

$$E(\tilde{a}_i)=\lambda a_i^-+(1-\lambda)a_i^+,\ i=1,2,\ \lambda\in[0,1] \quad (8)$$

where λ is an index that reflects the decision maker's risk bearing attitude.

The bigger the value $E(\tilde{a}_i)$, the greater the interval number $\tilde{a}_i$. In particular, if both $E(\tilde{a}_i)$ $(i=1,2)$ are equal, then we calculate the uncertainty indices of $\tilde{a}_i(i=1,2)$:

$$U(\tilde{a}_i)=a_i^+-a_i^-,\quad i=1,2 \quad (9)$$

The smaller the value $U(\tilde{a}_i)$, the less the uncertainty degree of $\tilde{a}_i$ is, and thus in this case, it is reasonable to stipulate that the greater the interval number $\tilde{a}_i$.

Based on both (8) and (9), we can compare any two interval numbers. Especially, if $E(\tilde{a}_1)=E(\tilde{a}_2)$ and $U(\tilde{a}_1)=U(\tilde{a}_2)$, then by (8) and (9), we have

$$\begin{cases}\lambda a_1^-+(1-\lambda)a_1^+=\lambda a_2^-+(1-\lambda)a_2^+\\ a_1^+-a_1^-=a_2^+-a_2^-\end{cases} \quad (10)$$

by which we get $a_1^-=a_2^-$ and $a_1^+=a_2^+$, i.e., $\tilde{a}_1=\tilde{a}_2$.

Let $\tilde{a}_i=[a_i^-,a_i^+]\,(i=1,2,...,n)$ be a collection of interval numbers, and $p,q\ge 0$, then we call

$$UB^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\tilde{a}_i^p\tilde{a}_j^q\right)^{\frac{1}{p+q}} \quad (11)$$

an uncertain Bonferroni mean (UBM) operator.

Based on the operations above, the UBM operator can be transformed into the following form:

$$UB^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$
$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(a_i^-\right)^p\left(a_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(a_i^+\right)^p\left(a_j^+\right)^q\right)^{\frac{1}{p+q}}\right] \quad (12)$$

Example 1. Given three interval numbers: $\tilde{a}_1=[10,15]$, $\tilde{a}_2=[8,10]$, and $\tilde{a}_3=[20,30]$. Without loss of generality, let $p=q=1$, then by (12) , we have

$$UB^{1,1}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)$$
$$=\left[\left(\frac{1}{6}(10\times8+10\times20+8\times20+8\times10+20\times10+20\times8)\right)^{\frac{1}{2}},\right.$$
$$\left.\left(\frac{1}{6}(15\times10+15\times30+10\times30+10\times15+30\times15+30\times10)\right)^{\frac{1}{2}}\right]$$
$$=[12.1,\ 17.3]$$

In the following, let us discuss some special cases of the UBM operator:

1) If $q=0$, then (12) reduces to

$$UB^{p,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$
$$=\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^-\right)^p\right)^{\frac{1}{p}},\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^+\right)^p\right)^{\frac{1}{p}}\right]=\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{a}_i^p\right)^{\frac{1}{p}} \quad (13)$$

which we call a generalized uncertain averaging operator.

1) If $p\to+\infty$, $q=0$, then (12) reduces to

$$\lim_{p\to+\infty}UB^{p,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$
$$=\left[\lim_{p\to+\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^-\right)^p\right)^{\frac{1}{p}},\ \lim_{p\to+\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^+\right)^p\right)^{\frac{1}{p}}\right]$$
$$=\left[\max_i\{a_i^-\},\ \max_i\{a_i^+\}\right] \quad (14)$$

1) If $p=1,\ q=0$, then (12) reduces to

$$UB^{1,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left[\frac{1}{n}\sum_{i}^{n}a_i^-,\frac{1}{n}\sum_{i}^{n}a_i^+\right]$$

$$=\frac{1}{n}\sum_{i=1}^{n}\tilde{a}_i \tag{15}$$

which is the uncertain averaging operator.

1) If $p\to 0,\ q=0$, then (12) reduces to

$$\lim_{p\to 0}UB^{p,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$

$$=\left[\lim_{p\to 0}\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^-\right)^p\right)^{\frac{1}{p}},\lim_{p\to 0}\left(\frac{1}{n}\sum_{i=1}^{n}\left(a_i^+\right)^p\right)^{\frac{1}{p}}\right]$$

$$=\left[\prod_{i=1}^{n}\left(a_i^-\right)^{\frac{1}{n}},\prod_{i=1}^{n}\left(a_i^+\right)^{\frac{1}{n}}\right]$$

$$=\left(\prod_{i=1}^{n}\tilde{a}_i\right)^{\frac{1}{n}} \tag{16}$$

which is the uncertain geometric mean operator.

2) If $p=q=1$, then (12) reduces to

$$UB^{1,1}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$

$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^-a_j^-\right)^2\right)^{\frac{1}{2}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^+a_j^+\right)^2\right)^{\frac{1}{2}}\right]$$

$$=\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\tilde{a}_i\tilde{a}_j\right)^2\right)^{\frac{1}{2}} \tag{17}$$

which we call an interrelated uncertain square mean operator.

The UBM operator has the following properties:

Theorem 1. Let $\tilde{a}_i=[a_i^-,a_i^+](i=1,2,...,n)$ be a collection of interval numbers, and $p,q\geq 0$, then

1) (**Idempotency**): $UB^{p,q}(\tilde{a},\tilde{a},...,\tilde{a})=\tilde{a}$, if $\tilde{a}_i=\tilde{a}$, for all i.

2) (**Monotonicity**): Let $\tilde{d}_i=[d_i^-,d_i^+]\,(i=1,2,...,n)$ be a collection of interval numbers, if $a_i^-\geq d_i^-$ and $a_i^+\geq d_i^+$, for all i, then $UB^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)\geq B^{p,q}(\tilde{d}_1,\tilde{d}_2,...,\tilde{d}_n)$.

3) (**Commutativity**): $UB^{p,q}(\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_n)=UB^{p,q}(\dot{\tilde{\alpha}}_1,\dot{\tilde{\alpha}}_2,...,\dot{\tilde{\alpha}}_n)$, for any permutation $(\dot{\tilde{\alpha}}_1,\dot{\tilde{\alpha}}_2,...,\dot{\tilde{\alpha}}_n)$ of $(\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_n)$.

4) (**Boundedness**): $\left[\min_i\{a_i^-\},\min_i\{a_i^+\}\right]\leq UB^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)\leq\left[\max_i\{a_i^-\},\max_i\{a_i^+\}\right]$.

Proof. 1) let $\tilde{a}=[a^-,a^+]$, then by (12), we have

$$UB^{p,q}(\tilde{a},\tilde{a},...,\tilde{a})$$

$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a^-\right)^p\left(a^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a^+\right)^p\left(a^+\right)^q\right)^{\frac{1}{p+q}}\right]$$

$$=\left[\left(\left(a^-\right)^{p+q}\right)^{\frac{1}{p+q}},\left(\left(a^+\right)^{p+q}\right)^{\frac{1}{p+q}}\right]=[a^-,a^+]=\tilde{a} \tag{18}$$

2) Since $a_i^-\geq d_i^-$ and $a_i^+\geq d_i^+$, for all i, then

$$UB^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$

$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^-\right)^p\left(a_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^+\right)^p\left(a_j^+\right)^q\right)^{\frac{1}{p+q}}\right]$$

$$\geq\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(d_i^-\right)^p\left(d_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(d_i^+\right)^p\left(d_j^+\right)^q\right)^{\frac{1}{p+q}}\right]$$

$$=UB^{p,q}(\tilde{d}_1,\tilde{d}_2,...,\tilde{d}_n) \tag{19}$$

3) Let $\dot{\tilde{a}}_i=[\dot{a}_i^-,\dot{a}_i^+]$ $(i=1,2,...,n)$, then

$$UB^{p,q}(\dot{\tilde{a}}_1,\dot{\tilde{a}}_2,...,\dot{\tilde{a}}_n)$$

$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\dot{a}_i^-\right)^p\left(\dot{a}_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\dot{a}_i^+\right)^p\left(\dot{a}_j^+\right)^q\right)^{\frac{1}{p+q}}\right] \tag{20}$$

Since $(\dot{\tilde{\alpha}}_1,\dot{\tilde{\alpha}}_2,...,\dot{\tilde{\alpha}}_n)$ is a permutation of $(\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_n)$, then by (12) and (20), we know that

$$\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\dot{a}_i^-\right)^p\left(\dot{a}_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\dot{a}_i^+\right)^p\left(\dot{a}_j^+\right)^q\right)^{\frac{1}{p+q}}\right]$$

$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^-\right)^p\left(a_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^+\right)^p\left(a_j^+\right)^q\right)^{\frac{1}{p+q}}\right] \tag{21}$$

i.e., $UB^{p,q}(\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_n)=UB^{p,q}(\dot{\tilde{\alpha}}_1,\dot{\tilde{\alpha}}_2,...,\dot{\tilde{\alpha}}_n)$.

4) By (12), we have

$$UB^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$$

$$=\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^-\right)^p\left(a_j^-\right)^q\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(a_i^+\right)^p\left(a_j^+\right)^q\right)^{\frac{1}{p+q}}\right]$$

$$\leq \left[\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\max_i \{a_i^-\} \right)^p \left(\max_i \{a_i^-\} \right)^q \right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\max_i \{a_i^+\} \right)^p \left(\max_i \{a_i^+\} \right)^q \right)^{\frac{1}{p+q}} \right]$$

$$= \left[\max_i \{a_i^-\}, \max_i \{a_i^+\} \right] \tag{22}$$

Similarly, we can prove $UB^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \geq \left[\min_i \{a_i^-\}, \min_i \{a_i^+\} \right]$.

As the input data usually come from different sources, and each datum has own importance, thus each datum should be assigned a weight. In this case, we shall consider the weighted form of the UBM operator.

Let $\tilde{a}_i = [a_i^-, a_i^+] (i = 1,2,...,n)$ be a collection of interval numbers, each $\tilde{a}_i$ has the weight w_i, satisfying $w_i \geq 0 \ \ (i = 1,2,...,n)$ and $\sum_{i=1}^{n} w_i = 1$. Then we call

$$UB_w^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\frac{1}{\theta} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (w_i \tilde{a}_i)^p (w_j \tilde{a}_j)^q \right)^{\frac{1}{p+q}} \tag{23}$$

a weighted uncertain Bonferroni mean (WUBM) operator, where

$$\theta = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (w_i)^p (w_j)^q \tag{24}$$

Based on the operations of interval numbers, the WUBM operator (23) can be further written as:

$$UB_w^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$$

$$= \left[\left(\frac{1}{\theta} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(w_i a_i^- \right)^p \left(w_j a_j^- \right)^q \right)^{\frac{1}{p+q}}, \left(\frac{1}{\theta} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(w_i a_i^+ \right)^p \left(w_j a_j^+ \right)^q \right)^{\frac{1}{p+q}} \right] \tag{25}$$

In the case where $w = (1/n, 1/n, ..., 1/n)^T$, (24) reduces to

$$\theta = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (w_i)^p (w_j)^q = n(n-1) \left(\frac{1}{n} \right)^{p+q} \tag{26}$$

and then (25) can be transformed to

$$UB_w^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$$

$$= \left[\left(\frac{1}{\theta} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{1}{n} a_i^- \right)^p \left(\frac{1}{n} a_j^- \right)^q \right)^{\frac{1}{p+q}}, \left(\frac{1}{\theta} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{1}{n} a_i^+ \right)^p \left(\frac{1}{n} a_j^+ \right)^q \right)^{\frac{1}{p+q}} \right]$$

$$= \left[\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(a_i^- \right)^p \left(a_j^- \right)^q \right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(a_i^+ \right)^p \left(a_j^+ \right)^q \right)^{\frac{1}{p+q}} \right]$$

$$= UB^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \tag{27}$$

which reduces to the UBM operator.

With the WUBM operator, let us give a simple approach to multi-criteria decision making under uncertainty:

Step 1. Let $X = \{x_1, x_2, .., x_n\}$ and $\Omega = \{c_1, c_2, .., c_m\}$ be the sets of alternatives and criteria respectively. Each criterion has a weight w_i, with $w_i \geq 0 \ \ (i = 1,2,...,m)$, and $\sum_{i=1}^{m} w_i = 1$. The performance of the alternative x_j with respect to the criterion c_i is described by a value range $\tilde{a}_{ij} = [a_{ij}^-, a_{ij}^+]$, which is listed in the decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$. In general, there are two types of criteria, i.e., benefit criteriaand cost criteria. We may normalize the matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ into the matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$ by the formulae:[22]

$$\begin{cases} r_{ij}^- = a_{ij}^- \Big/ \sum_{k=1}^{m} a_{ik}^+ \\ r_{ij}^+ = a_{ij}^+ \Big/ \sum_{k=1}^{m} a_{ik}^- \end{cases}, \quad \textit{for benefit criterion } c_i \tag{28}$$

$$\begin{cases} r_{ij}^- = \left(1/a_{ij}^+\right) \Big/ \sum_{k=1}^{m} \left(1/a_{ij}^-\right) \\ r_{ij}^+ = \left(1/a_{ij}^-\right) \Big/ \sum_{k=1}^{m} \left(1/a_{ij}^+\right) \end{cases}, \quad \textit{for } \cos t \textit{ criterion } c_i \tag{29}$$

where $\tilde{r}_{ij} = [r_{ij}^-, r_{ij}^+]$, $i = 1,2,...,m;\ j = 1,2,...,n$.

Step 2. Utilize the WUBM operator (25) (for the sake of intuitiveness and simplicity, in general, we take $p = q = 1$):

$$\tilde{r}_j = [r_j^-, r_j^+] = UB_w^{p,q}(\tilde{r}_{1j}, \tilde{r}_{2j}, ..., \tilde{r}_{mj}) \tag{30}$$

to aggregate all the performance values $\tilde{r}_{ij} (i = 1,2,...,m)$ of the j th column, and get the overall performance value $\tilde{r}_j$ corresponding to the alternative x_j.

Step 3. Utilize (8) and (9) to rank the overall performance values $\tilde{r}_j \ (j = 1,2,...,n)$, and by which we rank and select the alternatives $x_j \ \ (j = 1,2,...,n)$ following the principle that the greater the value $\tilde{r}_j$, the better the alternative x_j.

The prominent characteristic of the above approach is that it utilizes the WUBM operator to fuse the performance values of the alternatives, which can capture the interrelationship of the individual criteria.

Now we provide a numerical example to illustrate the application of the above approach:

Example 2. Robots are used extensively by many advanced manufacturing companies to perform dangerous and/or menial tasks.[22,23] The selection of a robot is an important function for these companies because improper selection of the robots may adversely affect their profitability. A manufacturing company intends to select a robot from five robots x_j $(j = 1,2,3,4,5)$. The following four criteria $c_i (i = 1,2,3,4)$ (whose weight vector is $w = (0.2, 0.3, 0.4, 0.1)^T$) have to be considered:

1) c_1: Velocity (m/s) which is the maximum speed the arm can achieve.

2) c_2: Load capacity (kg) which is the maximum weight a robot can lift.

3) c_3: Purchase, installation and training costs ($1000).

4) c_4: Repeatability (mm) which is a robot's ability to repeatedly return to a fixed position. The mean deviation from that position is a measure of a robot's repeatability.

Among these criteria, c_1 and c_2 are of benefit type, c_3 and c_4 are of cost type. The decision information about robots is listed in Table 1, and the normalized decision information by using (28) and (29) is listed in Table 2 (adopted from Ref.22).

Table 1. Uncertain decision matrix $\tilde{A}$

	x_1	x_2	x_3	x_4	x_5
c_1	[1.8, 2.0]	[1.4, 1.6]	[0.8, 1.0]	[1.0, 1.2]	[0.9, 1.1]
c_2	[90, 95]	[80, 85]	[65, 70]	[85, 90]	[70, 80]
c_3	[9.0, 9.5]	[5.5, 6.0]	[4.0, 4.5]	[9.5, 10]	[9.0, 10]
c_4	[0.45,0.50]	[0.30,0.40]	[0.20,0.25]	[0.25,0.30]	[0.35,0.40]

Table 2. Normalized uncertain decision matrix $\tilde{R}$

	x_1	x_2	x_3	x_4	x_5
c_1	[0.26, 0.34]	[0.20, 0.27]	[0.12, 0.17]	[0.14, 0.20]	[0.13, 0.19]
c_2	[0.21, 0.24]	[0.19, 0.22]	[0.15, 0.18]	[0.20, 0.23]	[0.17, 0.21]
c_3	[0.14, 0.16]	[0.22, 0.26]	[0.29, 0.36]	[0.13, 0.15]	[0.13, 0.16]
c_4	[0.11, 0.16]	[0.14, 0.23]	[0.23, 0.35]	[0.19, 0.28]	[0.14, 0.20]

Here we employ the WUBM operator (30) (let $p = q = 1$) to aggregate $\tilde{r}_{ij} (i = 1,2,3,4)$, and get the overall performance value $\tilde{r}_j$ of the robot x_j. Since

$$\theta = \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \left(w_i w_j \right)$$

$$= (0.2\times0.3 + 0.2\times0.4 + 0.2\times0.1 + 0.3\times0.4 + 0.3\times0.1 + 0.4\times0.1)\times2$$
$$= 0.70$$

then

$$\tilde{r}_1 = UB_w^{1,1}(\tilde{r}_{11}, \tilde{r}_{21}, \tilde{r}_{31}, \tilde{r}_{41})$$
$$= \left[\left(\frac{1}{0.7} \big((0.2\times0.26)\times(0.3\times0.21) + (0.2\times0.26)\times(0.4\times0.14) + (0.2\times0.26)\times(0.1\times0.11) + (0.3\times0.21)\times(0.4\times0.14) + (0.3\times0.21)\times(0.1\times0.11) + (0.4\times0.14)\times(0.1\times0.11) \big)\times2 \right)^{\frac{1}{2}}, \left(\frac{1}{0.7} \big((0.2\times0.34)\times(0.3\times0.24) + (0.2\times0.34)\times(0.4\times0.16) + (0.2\times0.34)\times(0.1\times0.16) + (0.3\times0.24)\times(0.4\times0.16) + (0.3\times0.24)\times(0.1\times0.16) + (0.4\times0.16)\times(0.1\times0.16) \big)\times2 \right)^{\frac{1}{2}} \right]$$
$$= [0.182, 0.221]$$

Similarly,

$$\tilde{r}_2 = [0.196, 0.246],\ \tilde{r}_3 = [0.195, 0.254]$$
$$\tilde{r}_4 = [0.160, 0.200],\ \tilde{r}_5 = [0.143, 0.186]$$

Using (8), we calculate the expected values of $\tilde{r}_j (j = 1,2,3,4,5)$:

$$E(\tilde{r}_1) = 0.221 - 0.039\lambda,\ E(\tilde{r}_2) = 0.246 - 0.050\lambda$$
$$E(\tilde{r}_3) = 0.254 - 0.059\lambda,\ E(\tilde{r}_4) = 0.200 - 0.040\lambda$$
$$E(\tilde{r}_5) = 0.186 - 0.043\lambda$$

Then by analyzing the parameter λ, we have

1) If $0 \le \lambda < \frac{8}{9}$, then

$$E(\tilde{r}_3) > E(\tilde{r}_2) > E(\tilde{r}_1) > E(\tilde{r}_4) > E(\tilde{r}_5)$$

Thus, $\tilde{r}_3 > \tilde{r}_2 > \tilde{r}_1 > \tilde{r}_4 > \tilde{r}_5$, by which we get the ranking of the robots:

$$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_5$$

2) If $\frac{8}{9} < \lambda \le 1$, then

$$E(\tilde{r}_2) > E(\tilde{r}_3) > E(\tilde{r}_1) > E(\tilde{r}_4) > E(\tilde{r}_5)$$

Thus, $\tilde{r}_2 > \tilde{r}_3 > \tilde{r}_1 > \tilde{r}_4 > \tilde{r}_5$, by which we get the ranking of the robots:

$$x_2 \succ x_3 \succ x_1 \succ x_4 \succ x_5$$

3) If $\lambda = \frac{8}{9}$, then

$$E(\tilde{r}_2) = E(\tilde{r}_3) > E(\tilde{r}_1) > E(\tilde{r}_4) > E(\tilde{r}_5)$$

In this case, we utilize (9) to calculate the uncertainty indices of $\tilde{r}_2$ and $\tilde{r}_3$:

$$U(\tilde{r}_2) = 0.246 - 0.196 = 0.050$$
$$U(\tilde{r}_3) = 0.254 - 0.195 = 0.059$$

Since $U(\tilde{r}_2) < U(\tilde{r}_3)$, then $\tilde{r}_2 > \tilde{r}_3$. In this case, $\tilde{r}_2 > \tilde{r}_3 > \tilde{r}_1 > \tilde{r}_4 > \tilde{r}_5$, therefore, the ranking of the robots is

$$x_2 \succ x_3 \succ x_1 \succ x_4 \succ x_5$$

From the analysis above, it is clear that the ranking of the robots maybe different as we change the parameter λ. When $\frac{8}{9} \le \lambda \le 1$, the robot x_2 is the best choice, while the robot x_3 is the second best one. But as $0 \le \lambda < \frac{8}{9}$, the ranking of x_2 and x_3 is reversed, i.e., the robot x_3 ranks first, while the robot x_2 ranks second. However, the ranking of the other robots $x_j (j = 1,4,5)$ keeps unchanged, i.e., $x_1 \succ x_4 \succ x_5$, for any $\lambda \in [0,1]$.

4. The UBM operators combined with the OWA operator and Choquet integral

In this section, we extend Yager' results[18] to uncertain situations by only considering the case where the parameters $p = q = 1$ in the UBM operator.

Let $\tilde{a}_i = [a_i^-, a_i^+]\ (i = 1,2,...,n)$ be a collection of interval numbers, then from (11), it yields

$$UB^{1,1}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \tilde{a}_i \tilde{a}_j \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{n} \sum_{i=1}^{n} \tilde{a}_i \left(\frac{1}{n-1} \sum_{\substack{j=1 \\ j \ne i}}^{n} \tilde{a}_j \right) \right)^{\frac{1}{2}} \quad (31)$$

For convenience, we denote $UB^{1,1}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$ as $UB(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$, and let $\tilde{\beta}_i = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \ne i}}^{n} \tilde{a}_j$, which is the uncertain average of all the interval numbers $\tilde{a}_j\ (j \ne i)$. Then (31) can be denoted as:

$$UB(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \tilde{a}_i \tilde{\beta}_i \right)^{\frac{1}{2}} \quad (32)$$

Suppose that $\tilde{v}^i$ is the $n-1$ tuple $(\tilde{a}_1, ..., \tilde{a}_{i-1}, \tilde{a}_{i+1}, ..., \tilde{a}_n)$. An uncertain ordered weighted averaging (UOWA) operator of dimension $n-1$ can be defined as:

$$UOWA_{\omega}(\tilde{v}^i) = \sum_{k=1}^{n-1} \omega_k \tilde{a}_{\sigma_i(k)} = \left[\sum_{k=1}^{n-1} \omega_k \tilde{a}^-_{\sigma_i(k)}, \sum_{k=1}^{n-1} \omega_k \tilde{a}^+_{\sigma_i(k)} \right] \quad (33)$$

where $\tilde{a}_{\sigma_i(k)} = [a^-_{\sigma_i(k)}, a^+_{\sigma_i(k)}]$ is the k th largest interval number in the tuple $\tilde{v}^i$, $\omega = (\omega_1, \omega_2, ..., \omega_{n-1})^T$ is the weighting vector associated with the UOWA operator, $\omega_k \ge 0$ and $\sum_{k=1}^{n-1} \omega_k = 1$.

If we replace the uncertain average $\tilde{\beta}_i$ in (32) with the UOWA aggregation of all $\tilde{a}_j\ (j \ne i)$, then from (33), it follows that

$$UB-OWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \tilde{a}_i\, UOWA_{\omega}(\tilde{v}^i) \right)^{\frac{1}{2}} \quad (34)$$

which we call a UBM-OWA operator. Especially, if $\omega = (1/n-1, 1/n-1, ..., 1/n-1)^T$, then (34) reduces to the UBM operator.

If we take the weights of the data into account, and let $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of $\tilde{a}_i$ ($i = 1,2,...,n$), with $w_i \ge 0\ (i = 1,2,...,n)$ and $\sum_{i=1}^{n} w_i = 1$. Then (34) can be generalized as:

$$UB-OWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\sum_{i=1}^{n} w_i \tilde{a}_i\, UOWA_{\omega}(\tilde{v}^i) \right)^{\frac{1}{2}} \quad (35)$$

In particular, if $w = (1/n, 1/n, ..., 1/n)^T$, then (35) reduces to (34).

Example 3. Let $\tilde{a}_1 = [3,5]$, $\tilde{a}_2 = [1,2]$, $\tilde{a}_3 = [7,9]$ be three interval numbers, $w = (0.3, 0.4, 0.3)^T$ be the weight vector of $\tilde{a}_i (i = 1,2,3)$, and $\omega = (0.6, 0.4)^T$ be the weighting vector associated with the UOWA operator of dimension 2.

Since $\tilde{a}_3 > \tilde{a}_1 > \tilde{a}_3$, then we first calculate the values of the $UOWA_{\omega}(\tilde{v}^i)\ (i = 1,2,3)$:

$$UOWA_{\omega}(\tilde{v}^1) = UOWA_{\omega}(\tilde{a}_2, \tilde{a}_3) = \omega_1 \tilde{a}_3 + \omega_2 \tilde{a}_2$$
$$= 0.6 \times [7,9] + 0.4 \times [1,2] = [4.6, 6.2]$$
$$UOWA_{\omega}(\tilde{v}^2) = UOWA_{\omega}(\tilde{a}_1, \tilde{a}_3) = \omega_1 \tilde{a}_3 + \omega_1 \tilde{a}_1$$
$$= 0.6 \times [7,9] + 0.4 \times [3,5] = [5.4, 7.4]$$
$$UOWA_{\omega}(\tilde{v}^3) = UOWA_{\omega}(\tilde{a}_1, \tilde{a}_2) = \omega_1 \tilde{a}_1 + \omega_2 \tilde{a}_2$$
$$= 0.6 \times [3,5] + 0.4 \times [1,2] = [2.2, 3.8]$$

and then by (35), we have

$$UB-OWA(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \left(\sum_{i=1}^{3} w_i \tilde{a}_i\, UOWA_{\omega}(\tilde{v}^i) \right)^{\frac{1}{2}}$$

$$= \left(w_1 \tilde{a}_1\, UOWA_{\omega}(\tilde{v}^1) + w_2 \tilde{a}_2\, UOWA_{\omega}(\tilde{v}^2) + w_3 \tilde{a}_3\, UOWA_{\omega}(\tilde{v}^3) \right)^{\frac{1}{2}}$$

$$= \left(0.3 \times [3,5] \times [4.6, 6.2] + 0.4 \times [1,2] \times [5.4, 7.4] + 0.3 \times [7,9] \times [2.2, 3.8] \right)^{\frac{1}{2}}$$

$$= [3.46, 4.86]$$

In what follows, let us consider how to combine the UBM operator with the well-known Choquet integral.

Let the criteria sets Ω, Ω^i and the monotonic set measure m_i over Ω^i be defined as in Section 2. In addition, let $\tilde{a}_{\sigma_i(1)}, \tilde{a}_{\sigma_i(2)}, \ldots, \tilde{a}_{\sigma_i(n-1)}$ be the ordered interval numbers in $\tilde{v}^i$, such that $\tilde{a}_{\sigma_i(k-1)} \geq \tilde{a}_{\sigma_i(k)}$, $k = 2,3,\ldots,n-1$, and let $\tilde{B}_{\sigma_i(j)} = \{\tilde{a}_{\sigma_i(k)} \mid k \leq j\}$, when $j \geq 1$ and $\tilde{B}_{\sigma_i(0)} = \phi$. Then the Choquet integral of $\tilde{v}^i$ with respect to m_i can be defined as:

$$C_{m_i}(\tilde{v}^i) = \sum_{j=1}^{n-1} \tilde{a}_{\sigma_i(j)} \left(m_i(\tilde{B}_{\sigma_i(j)}) - m_i(\tilde{B}_{\sigma_i(j-1)}) \right) \tag{36}$$

by which we define

$$UB-CHOQ(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \tilde{a}_i \, C_{m_i}(\tilde{v}^i) \right)^{\frac{1}{2}} \tag{37}$$

as an uncertain Bonferroni Choquet (UBM-CHOQ) operator.

If we take the weight w_i of each $\tilde{a}_i$ into account, then by (37), we have

$$UB-CHOQ(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\sum_{i=1}^{n} w_i \tilde{a}_i \, C_{m_i}(\tilde{v}^i) \right)^{\frac{1}{2}} \tag{38}$$

In the special case where $w = (1/n, 1/n, \ldots, 1/n)^T$, (38) reduces to (37).

To illustrate the UB-CHOQ operator, we give the following example:

Example 4. Assume we have three criteria $c_i (i = 1,2,3)$, whose weight vector is $w = (0.5, 0.3, 0.2)^T$, the performance of an alternative x with respect to the criteria $c_i (i = 1,2,3)$ is described by the interval numbers: $\tilde{a}_1 = [3,4]$, $\tilde{a}_2 = [5,7]$, $\tilde{a}_3 = [4,6]$. Let

$$m_1(\phi) = m_2(\phi) = m_3(\phi) = 0, \; m_1(\{\tilde{a}_2\}) = m_3(\{\tilde{a}_2\}) = 0.3$$
$$m_1(\{\tilde{a}_3\}) = m_2(\{\tilde{a}_3\}) = 0.5, \; m_2(\{\tilde{a}_1\}) = m_3(\{\tilde{a}_1\}) = 0.6$$
$$m_1(\{\tilde{a}_2, \tilde{a}_3\}) = m_2(\{\tilde{a}_3, \tilde{a}_1\}) = m_3(\{\tilde{a}_2, \tilde{a}_1\}) = 1$$

Then by (36), we have

$$C_{m_1}(\tilde{v}^1) = \sum_{j=1}^{2} \tilde{a}_{\sigma_1(j)} \left(m_1(\tilde{B}_{\sigma_1(j)}) - m_1(\tilde{B}_{\sigma_1(j-1)}) \right)$$
$$= \tilde{a}_2 \times \left(m_1(\{\tilde{a}_2\}) - m_1(\phi) \right) + \tilde{a}_3 \times \left(m_1(\{\tilde{a}_2, \tilde{a}_3\}) - m_1(\{\tilde{a}_2\}) \right)$$
$$= [5,7] \times (0.3 - 0) + [4,6] \times (1 - 0.3)$$
$$= [4.3, 6.3]$$

$$C_{m_2}(\tilde{v}^2) = \sum_{j=1}^{2} \tilde{a}_{\sigma_2(j)} \left(m_2(\tilde{B}_{\sigma_2(j)}) - m_2(\tilde{B}_{\sigma_2(j-1)}) \right)$$
$$= \tilde{a}_3 \times \left(m_2(\{\tilde{a}_3\}) - m_2(\phi) \right) + \tilde{a}_1 \times \left(m_2(\{\tilde{a}_3, \tilde{a}_1\}) - m_2(\{\tilde{a}_3\}) \right)$$
$$= [4,6] \times (0.5 - 0) + [3,4] \times (1 - 0.5)$$
$$= [3.5, 5.0]$$

$$C_{m_3}(\tilde{v}^3) = \sum_{j=1}^{2} \tilde{a}_{\sigma_3(j)} \left(m_3(\tilde{B}_{\sigma_3(j)}) - m_3(\tilde{B}_{\sigma_3(j-1)}) \right)$$
$$= \tilde{a}_2 \times \left(m_3(\{\tilde{a}_2\}) - m_3(\phi) \right) + \tilde{a}_1 \times \left(m_3(\{\tilde{a}_2, \tilde{a}_1\}) - m_1(\{\tilde{a}_2\}) \right)$$
$$= [5,7] \times (0.3 - 0) + [3,4] \times (1 - 0.3)$$
$$= [3.6, 4.9]$$

and then from (38), it yields

$$UB-CHOQ(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \left(\sum_{i=1}^{3} w_i \tilde{a}_i \, C_{m_i}(\tilde{v}^i) \right)^{\frac{1}{2}}$$
$$= \left(0.5 \times [3,4] \times [4.3, 6.3] + 0.3 \times [5,7] \times [3.5, 5.0] + 0.2 \times [4,6] \times [3.6, 4.9] \right)^{\frac{1}{2}}$$
$$= \left([6.45, 12.60] + [5.25, 10.50] + [2.88, 5.88] \right)^{\frac{1}{2}}$$
$$= [14.58, 28.98]^{\frac{1}{2}}$$
$$= [3.82, 5.38]$$

5. Conclusions

We have investigated the Bonferroni mean under uncertain situations, and developed an uncertain Bonferroni mean (UBM) operator that can capture the interrelationship between the individual uncertain data. We have given a weighting form of the UBM operator, in which the importance of each datum can be taken into account, and then developed a simple approach based the weighted UBM operator to multi-criteria decision making. Furthermore, based on the well-known ordered weighted averaging (OWA) operator and Choquet integral, we have proposed some combined UBM operators, and detailedly illustrated their operational processes with some numerical examples. In further research, it is necessary and meaningful to give the applications of these operators to the other fields such as pattern recognition, fuzzy cluster analysis and uncertain programming, etc.

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