A New Minkowski Distance Based on Induced Aggregation Operators

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Received: 25-05-2009
Accepted: 01-12-2010

Abstract

The Minkowski distance is a distance measure that generalizes a wide range of distances such as the Hamming and the Euclidean distance. In this paper, we develop a generalization of the Minkowski distance by using the induced ordered weighted averaging (IOWA) operator. We call it the induced Minkowski OWA distance (IMOWAD) or induced generalized OWA distance (IGOWAD) operator. Then, we are able to obtain a wide range of distance measures that includes the Minkowski distance, the Minkowski OWA distance (MOWAD), and the induced OWA distance (IOWAD). We also present a further generalization by using quasi-arithmetic means. We end the paper with a numerical example of the new approach.

Keywords: Minkowski distance, Aggregation operators, IOWA operator, Decision making.

1. Introduction

The Minkowski distance is one of the main distance measures because it generalizes a wide range of other distances such as the Hamming distance and the Euclidean distance. Often, when calculating distances, we want an average result of all the individual distances. We call this the normalization process. In the literature, we find mainly three types of normalized distances. The first one is when we use the arithmetic mean and it is known as the normalized Minkowski distance (NMD). The second one is when we use the weighted average (WA) and it is known as the weighted Minkowski distance (WMD). The third one is when we use the ordered weighted averaging (OWA) operator[1-24] and it is known as the Minkowski ordered weighted averaging distance (MOWAD) operator.[8,11] Note that the MOWAD includes the NMD and the WMD as special cases.

Sometimes, when normalizing the Minkowski distance with the OWA operator, it would be interesting to consider a more general formulation of the reordering process. A very useful technique for doing so is the induced OWA (IOWA) operator.[17,21] The IOWA operator provides a parameterized family of aggregation operators such as the maximum, the minimum, the average and the OWA operator. Thus, we are able to use complex reordering processes in the aggregation step of the IOWA operator. This can be useful in a lot of situations such as in decision making problems[25-28] where we may consider complex attitudinal characters of the decision maker instead of simply considering the degree of optimism or pessimism. Recently, Merigó and Gil-Lafuente[12] suggested a more general formulation of the IOWA operator by using generalized and quasi-arithmetic means. They called these new aggregation operators, the induced generalized OWA (IGOWA) operator and the induced Quasi-OWA (Quasi-IOWA) operator. They generalize a wide range of aggregation operators including the IOWA operator, the OWA operator, the average and the weighted average.

In this paper, we suggest a new type of distance measure consisting in normalizing the Minkowski distance by using the IOWA operator. Then, the normalization developed will be able to reflect complex attitudinal characters. We will call this generalization as the induced Minkowski OWA distance (IMOWAD) operator. The main advantage of this operator is that it generalizes a wide range of distances such as the NMD, the WMD, the MOWAD, the induced OWA distance (IOWAD), the induced Euclidean OWA distance (IEOWAD) and a lot of other particular cases. Another advantage of the IMOWAD operator is that it is able to deal with complex attitudinal characters (or complex degrees of orness) in the aggregation process. Therefore, we are able to deal with more complex situations more close to the real world.

We further generalize the IMOWAD operator by using quasi-arithmetic means. As a result, we get the Quasi-IOWAD operator. It is a more complete generalization because it includes the IMOWAD operator and a lot of other situations. We also develop an application of the new approach in a decision making problem about selection of investments. We see that depending on the particular type of IMOWAD operator used, the results may lead to different decisions.

This paper is organized as follows. In Section 2, we briefly review some basic concepts about the Minkowski distance, the IOWA and the IGOWA operator. In Section 3 we present the IMOWAD operator. Section 4 analyzes different families of IMOWAD operators. In Section 5 we present the Quasi-IOWAD operator. Section 6 analyzes the applicability of the new approach and Section 7 develops a numerical example of the new generalization. Finally, in Section 8 we summarize the main conclusions of the paper.

2. Preliminaries

In this Section, we briefly review the Minkowski distance, the IOWA operator and the IGOWA operator.

2.1. *Normalized Minkowski Distance*

The normalized Minkowski distance is a distance measure[29-34] that generalizes a wide range of distances such as the normalized Hamming distance and the normalized Euclidean distance. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, etc.

In order to define the Minkowski distance, first, we define a distance measure. Essentially, a distance measure has to accomplish the following properties.

1) Non-negativity: $D(A_1, A_2) \geq 0$.
2) Commutativity: $D(A_1, A_2) = D(A_2, A_1)$.
3) Reflexivity: $D(A_1, A_1) = 0$.
4) Triangle inequality: $D(A_1, A_2) + D(A_2, A_3) \geq D(A_1, A_3)$.

The normalized Minkowski distance can be formulated for two sets *A* and *B* as follows.

Definition 1. *A normalized Minkowski distance of dimension n is a mapping* d_m*:* $R^n \times R^n \rightarrow R$ *such that*:

$$d_m(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} | a_i - b_i |^{\lambda} \right)^{1/\lambda} \tag{1}$$

where a_i *and* b_i *are the ith arguments of the sets A and B and* λ *is a parameter such that* $\lambda \in (-\infty, \infty)$.

If we give different values to the parameter λ, we can obtain a wide range of special cases. For example, if $\lambda = 1$, we obtain the normalized Hamming distance (NHD). If $\lambda = 2$, the normalized Euclidean distance (NED).

Sometimes, when normalizing the Minkowski distance, we prefer to give different weights to each individual distance. Thus, the distance is known as the weighted Minkowski distance. It can be defined as follows.

Definition 2. *A weighted Minkowski distance of dimension n is a mapping* d_{wm}*:* $R^n \times R^n \rightarrow R$ *that has an associated weighting vector W of dimension n with* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0, 1]$, *such that:*

$$d_{wm}(A, B) = \left(\sum_{i=1}^{n} w_i \, | a_i - b_i |^{\lambda} \right)^{1/\lambda} \tag{2}$$

where a_i *and* b_i *are the ith arguments of the sets A and B and* λ *is a parameter such that* $\lambda \in (-\infty, \infty)$.

In this case, we can also obtain a wide range of special cases by using different values in the parameter λ. For example, if $\lambda = 1$, we obtain the weighted

Hamming distance (WHD). If $\lambda = 2$, the weighted Euclidean distance (WED).

2.2. *Induced OWA Operator*

The IOWA operator was introduced by Yager and Filev[21] and it is an extension of the OWA operator. Its main difference is that the reordering step is not developed with the values of the arguments. In this case, the reordering step is developed with order inducing variables. It can be defined as follows.

Definition 3. *An IOWA operator of dimension n is a mapping f: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:*

$$f(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{3}$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

The IOWA operator includes the OWA operator as a particular case and a lot of other situations such as the maximum, the minimum and the average. Note that it is possible to distinguish between the descending IOWA (DIOWA) operator and the ascending IOWA (AIOWA) operator.

2.3. *Induced Generalized OWA Operator*

The IGOWA operator was introduced in Ref. 12 and it represents a generalization of the IOWA operator by using generalized means. Therefore, it is possible to include in the same formulation, different types of induced aggregation operators such as the IOWA operator or the induced OWG (IOWG) operator. It can be defined as follows.

Definition 4. *An IGOWA operator of dimension n is a mapping IGOWA: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then*:

$$f(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \tag{4}$$

where b_j is the a_i value of the IGOWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, a_i is the argument variable and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

As we can see, if $\lambda = 1$, we get the IOWA operator. If $\lambda = 0$, the IOWG operator and if $\lambda = 2$, the IOWQA operator. Note that it is possible to further generalize the IGOWA operator by using quasi-arithmetic means. The result is the Quasi-IOWA operator. For further reading on the IGOWA and the Quasi-IOWA, refer to Ref. 12.

3. The Induced Minkowski OWA Distance

The IMOWAD operator is a distance measure that uses the IOWA operator in the normalization process of the Minkowski distance. Then, the reordering of the individual distances is developed with order inducing variables. For two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, it can be defined as follows.

Definition 5. *An IMOWAD operator is a mapping f: $R^n \times R^n \times R^n \rightarrow R$ that has an associated weighting vector W with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \tag{5}$$

where b_j is the $|x_i - y_i|$ value of the IMOWAD triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, $|x_i - y_i|$ is the argument variable represented in the form of individual distances and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Example 1. Assume two sets $A = (0.2, 0.4, 0.7, 0.3)$ and $B = (0.9, 0.4, 0.7, 0.2)$, and $\lambda = 1$. Assume that both sets have the same order-inducing variables $U = (6, 8, 3, 7)$. And assume the following weighting vector $W = (0.3, 0.3, 0.2, 0.2)$. Then, the IMOWAD can be calculated as follows.

<u>IMOWAD triplets</u>

$\langle 6, 0.2, 0.9\rangle = \langle 6, 0.7\rangle$

$\langle 8, 0.4, 0.4\rangle = \langle 8, 0\rangle$

$\langle 3, 0.7, 0.7\rangle = \langle 3, 0\rangle$

$\langle 7, 0.3, 0.2\rangle = \langle 7, 0.1\rangle$

$$f = 0.3 \times 0 + 0.3 \times 0.1 + 0.2 \times 0.7 + 0.2 \times 0 = 0.17.$$

Note that sometimes, as in fuzzy set theory, it is better to use in the definition, a mapping f: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$.

A further interesting aspect to consider is the reordering process of the information.[35] Usually, we reorder the IMOWAD according to the values of the u_i, but it is also possible to adapt them to the initial positions of the arguments. That is:

$$f(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \left(\sum_{i=1}^{n} |x_i - y_i|^{\lambda} w_i \right)^{1/\lambda} \quad (6)$$

where w_i is the ith weight w_j reordered according to the positions of the $|x_i - y_i|$ and using order-inducing variables u_i.

The IOWAD operator is commutative, monotonic, bounded and idempotent. These properties can be proved with the following theorems.

Theorem 1 (Commutativity). *Assume f is the IMOWAD operator, then*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle \quad (7)$$

where $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$ *is any permutation of the arguments* $(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle)$.

Proof. It is straightforward and thus omitted. □

Theorem 2 (Monotonicity). *Assume f is the IMOWAD operator, if* $|x_i - y_i| \geq |c_i - d_i|$, *for all i, then*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) \quad (8)$$

Proof. It is straightforward and thus omitted. □

Theorem 3 (Bounded). *Assume f is the IMOWAD operator, then*:

$$\min\{|x_i - y_i|\} \leq f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \max\{|x_i - y_i|\} \quad (9)$$

Proof. It is straightforward and thus omitted. □

Theorem 4 (Idempotency). *Assume f is the IMOWAD operator, if* $|x_i - y_i| = a$, *for all i, then*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = a \quad (10)$$

Proof. It is straightforward and thus omitted. □

Remark 1. Note that if $x_i = y_i$ for all $i \in [1, n]$, $f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = 0$. Note also that $f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f(\langle u_1, y_1, x_1\rangle, \ldots, \langle u_n, y_n, x_n\rangle)$.

Remark 2. Note that it is possible to distinguish between descending (DIMOWAD) and ascending (AIMOWAD) orders. The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DIMOWAD (or IMOWAD) operator and w^*_{n+1-j} the jth weight of the AIMOWAD operator.

Remark 3. If B is a vector corresponding to the ordered arguments b_j^{λ}, we shall call this the ordered argument vector and W^T is the transpose of the weighting vector, then, the IMOWAD operator can be expressed as:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(W^T B\right)^{1/\lambda} \quad (11)$$

Remark 4. Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IMOWAD operator can be expressed as:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\frac{1}{W} \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (12)$$

Some other interesting generalizations can be developed following.[36-39] Following,[38] we can develop the function IMOWAD operator that uses a generating function r for the order inducing variables such that, r: $I \to R$, being $I \subset R$ a closed interval $I = [a, b]$. Moreover, we use a more general representation by using also a generating function for the arguments such that, s: $R^m \to R$. Furthermore, we also use a weighting function f for the weighting vector. Thus, we obtain the function induced generalized mixture distance (IGMD) operator as follows. Note that in this definition we refer to the arguments as two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 6. *An IGMD operator of dimension n is a mapping IMD*: $R^n \times R^n \times R^n \to R$ *that has an associated a vector of weighting functions f, r*: $I \to]0, \infty[$, *is some positive continuous function, s*: $R^m \to R$, *such that*:

$$IGMD(\langle r_o(u_1), s_p(x_1), s_q(y_1)\rangle, \ldots, \langle r_o(u_n), s_p(x_n), s_q(y_n)\rangle) =$$

$$= \left(\frac{\sum_{j=1}^{n} f_j(s_y(b_j)) s_y(b_j)^{\lambda}}{\sum_{j=1}^{n} f_j(s_y(b_j))} \right)^{1/\lambda} \quad (13)$$

where $s_y(b_j)$ *is the* $|s_p(x_i) - s_q(y_i)|$ *value of the IGMD triplet* $\langle r_o(u_i), s_p(x_i), s_q(y_i)\rangle$ *having the jth largest* $r_o(u_i)$, u_i *is the order-inducing variable,* $|s_p(x_i) - s_q(y_i)|$ *is the argument variable represented in the form of individual distances; o, p and q indicates that each order-inducing variable and each argument is formed by using a different function and* λ *is a parameter such that* $\lambda \in (-\infty, \infty)$.

Following,[12] we can obtain a wide range of particular cases of the IGMD operator. For example:

- If $\lambda = 1$, we obtain the induced mixture distance (IMD) operator.
- If $\lambda = 2$, the induced quadratic mixture distance (IQMD) operator.
- If $\lambda = 3$, the induced cubic mixture distance (ICMD) operator.
- If $\lambda \to 0$, the induced geometric mixture distance operator
- If $\lambda = -1$, the induced harmonic mixture distance (IHMD) operator.

A further interesting extension consists in using infinitary aggregation operators.[36] In this case, we assume that there are an unlimited number of arguments that appear in the aggregation process. Note that $\sum_{j=1}^{\infty} w_j = 1$. By using the IMOWAD operator we get the infinitary IMOWAD (∞-IMOWAD) operator as follows.

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{\infty} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (14)$$

However, note that the reordering process is much more complex due to the fact that we never know which argument should go in the first or in the last position because we have an unlimited number of arguments. This problem can be partially solved by using some kind of special instructions in the order inducing variables such that we aggregate the information partially in order to obtain partial results. Further reading for the usual OWA, see. [36]

Note that a similar extension could be developed by using the IGMD operator, obtaining the ∞-IGMD operator.

A further interesting issue is the problem of ties in the reordering process of the order inducing variables. In order to solve this problem, we recommend to follow the policy explained in Ref. 21 about replacing the tied arguments by their average. Note that in this case, it would mean that we are replacing the tied arguments by their normalized Minkowski distance.

Note that in the analysis of the order-inducing variables of the IMOWAD operator, we should note that the values used can be drawn from any space, with the only requirement of having a linear ordering. Therefore, it is possible to use different kinds of attributes for the order-inducing variables that permit us, for example, to mix numbers with words in the aggregations.

Other factors that we can consider are the measures for characterizing a weighting vector and the type of aggregation it performs.[8,15] The first measure $\alpha(W)$, the degree of orness, is defined as:

$$\alpha(W) = \left(\sum_{j=1}^{n} \left(\frac{n-j}{n-1} \right)^{\lambda} w_j^* \right)^{1/\lambda} \quad (15)$$

Note that w_j* is the w_j weight of the IMOWAD aggregation ordered in descending order according to the values of the arguments $|x_i - y_i|$. It can be shown that $\alpha \in [0, 1]$. The more weight is located near the top of W, the closer α is to 1, while the more weight is located toward the bottom of W, the closer α is to 0. Note also that we can use the dual, that is: Andness = 1 – Orness.

The second measure[15] is called the entropy of dispersion of the weighting vector W. It is defined as:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (16)$$

For example, if $w_j = 1$ for some j, then $H(W) = 0$, and thus the least amount of information is used.

The divergence of W measures the divergence of the weights against the degree of orness:

$$Div(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W) \right)^2 \quad (17)$$

The balance operator measures the balance of the weights against the orness or the andness:

$$Bal(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1} \right) w_j \quad (18)$$

It can be shown that $Bal(W) \in [-1, 1]$. Note that for the optimistic criteria, $Bal(W) = 1$, and for the pessimistic criteria, $Bal(W) = -1$.

4. Families of IMOWAD Operators

In this Section, we analyze different particular cases of the IMOWAD operator. We distinguish between those families found in the parameter λ and those found in the weighting vector W.

4.1. *Analysing the parameter λ*

By looking to the parameter λ, we can find a wide range of distance measures such as the IOWAD, the EIOWAD, the induced ordered weighted geometric distance (IOWGD) operator, the induced ordered weighted harmonic averaging distance (IOWHAD) operator and a lot of other cases.

Remark 5. When $\lambda = 1$, we get the IOWAD operator.

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \quad (19)$$

Note that if $w_j = 1/n$, for all a_i, we get the NHD. The WHD is obtained if $u_i > u_{i+1}$, for all i, and the OWAD operator is obtained if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of $|x_i - y_i|$.

Remark 6. When $\lambda = 2$, we get the IEOWAD operator.

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^2 \right)^{1/2} \quad (20)$$

If $w_j = 1/n$, for all a_i, we get the NED. If $u_i > u_{i+1}$, for all i, we get the WED and if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of $|x_i - y_i|^2$, we get the EOWAD operator.

Remark 7. When $\lambda = 0$, we get the IOWGD operator.

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \prod_{j=1}^{n} b_j^{\,w_j} \quad (21)$$

Note that if $w_j = 1/n$, for all a_i, we get the normalized geometric distance and if $u_i > u_{i+1}$, for all i, the weighted geometric distance. If the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of $|x_i - y_i|^{w_j}$, we get the ordered weighted geometric distance operator (OWGD) operator.

Remark 8. When $\lambda = -1$, we get the IOWHAD operator.

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}} \quad (22)$$

Note that if $w_j = 1/n$, for all a_i, we get the normalized harmonic distance. If $u_i > u_{i+1}$, for all i, we get the weighted harmonic distance. If the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of $1/(w_j / |x_i - y_i|)$, we get the ordered weighted harmonic averaging distance operator (OWHAD) operator .

Remark 9. When $\lambda = 3$, we get the induced ordered weighted cubic averaging distance (IOWCAD) operator.

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^3 \right)^{1/3} \quad (23)$$

If $w_j = 1/n$, for all a_i, we get the normalized cubic distance (NCD). If $u_i > u_{i+1}$, for all i, we get the weighted cubic distance (WCD) and if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of $|x_i - y_i|^2$, we get the ordered weighted cubic averaging distance (OWCAD) operator.

Note that we could analyze other families by using different values in the parameter λ. Note also that it is possible to study these families individually in a similar way as it has been developed in Section 3.

4.2. *Analysing the weighting vector W*

By choosing a different manifestation of the weighting vector in the IMOWAD operator, we are able to obtain different types of distance aggregation operators. For example, we can obtain the maximum distance, the minimum distance, the NMD, the WMD and the MOWAD operator.

- The maximum distance is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\{|x_i - y_i|\}$.
- The minimum distance is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\{|x_i - y_i|\}$.
- More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we are using the step-IMOWAD operator.
- The NMD is found when $w_j = 1/n$, for all i.
- The WMD is obtained if $u_i > u_{i+1}$, for all i.
- The MOWAD operator is obtained if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of $|x_i - y_i|$.
- Note that the IGOWA operator[12] is also included as a particular case of IMOWAD operator. This situation appears when one of the sets of the IMOWAD operator is empty.

Remark 10. Other families of IMOWAD operators could be used. For more information, see.[8,16,19] For example, when $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-IMOWAD operator. Note that k and m must be positive integers such that $k + m - 1 \leq n$.

Remark 11. If $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$, we are using the olympic-IMOWAD. Note that it is possible to present a general form of the olympic-IMOWAD operator considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then, this general form becomes the usual olympic-IMOWAD.

Remark 12. Note that the IMOWAD-median and the weighted IMOWAD-median can also be used as a particular case of the IMOWAD. For the IMOWAD median, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the argument a_i with the $[(n + 1)/2]$th largest u_i. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest u_i. For the weighted IMOWAD median, we select the argument a_i that has the kth largest inducing variable u_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5. Note that if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, then, we get the IMOWAD-median and the weighted IMOWAD-median, respectively.

Remark 13. Another interesting family is the S-IMOWAD operator. It can be divided in three classes, the "orlike", the "andlike" and the generalized S-IMOWAD operator. The generalized S-IMOWAD operator is obtained when $w_p = (1/n)(1 - (\alpha + \beta)) + \alpha$, with $u_p = \text{Max}\{a_i\}$; $w_q = (1/n)(1 - (\alpha + \beta)) + \beta$, with $u_q = \text{Min}\{a_i\}$; and $w_j = (1/n)(1 - (\alpha + \beta))$ for all $j \neq p,q$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-IMOWAD operator becomes the "andlike" S-IMOWAD operator and if $\beta = 0$, it becomes the "orlike" S-IMOWAD operator.

Remark 14. A further interesting family that could be used is the centered-IMOWAD operator. An IMOWAD operator is a centered aggregation operator if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered-IMOWAD operator. Another particular situation of the centered-IMOWAD operator appears if we remove the third condition. We will refer to it as a non-inclusive centered-IMOWAD operator.

5. The Induced Quasi-OWAD Operator

The IMOWAD can be generalized by using quasi-arithmetic means in a similar way as it was done in Ref. 12. We call it the Quasi-IOWAD operator. It is defined as follows.

Definition 7. *A Quasi-IOWAD operator is a mapping f*: $R^n \times R^n \times R^n \rightarrow R$ *that has an associated weighting vector W with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *such that*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\left(b_{(j)}\right)\right) \quad (24)$$

where b_j is the $|x_i - y_i|$ value of the QIOWAD triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, $|x_i - y_i|$ is the argument variable represented in the form of individual distances, and g is the strictly continuous monotonic function.

As we can see, when $g(b) = b^\lambda$, then, the Quasi-IOWAD becomes the IMOWAD operator. Note that it is also possible to distinguish between descending (Quasi-DIOWAD) and ascending (Quasi-AIOWAD) orders.

Note that all the properties and particular cases commented in the IMOWAD operator are also applicable in the Quasi-IOWAD operator.

For example, we could mention the trigonometric IOWAD operator, the exponential IOWAD operator and the radical IOWAD operator.

The trigonometric IOWAD is obtained when $g_1(t) = \sin((\pi/2)\ t)$, $g_2(t) = \cos((\pi/2)\ t)$ and $g_3(t) = \tan((\pi/2)\ t)$ are the generating functions. Thus, the trigonometric IOWAD functions are:

$$f(\langle u_1, x_1, y_1\rangle,\ldots,\langle u_n, x_n, y_n\rangle) = \frac{2}{\pi}\arcsin\left(\sum_{j=1}^{n} w_j \sin\left(\frac{\pi}{2} b_j\right)\right) \quad (25)$$

$$f(\langle u_1, x_1, y_1\rangle,\ldots,\langle u_n, x_n, y_n\rangle) = \frac{2}{\pi}\arccos\left(\sum_{j=1}^{n} w_j \cos\left(\frac{\pi}{2} b_j\right)\right) \quad (26)$$

$$f(\langle u_1, x_1, y_1\rangle,\ldots,\langle u_n, x_n, y_n\rangle) = \frac{2}{\pi}\arctan\left(\sum_{j=1}^{n} w_j \tan\left(\frac{\pi}{2} b_j\right)\right) \quad (27)$$

The exponential IOWAD is formed when $g(t) = \gamma^t$, if $\gamma \neq 1$, and $g(t) = t$, if $\gamma = 1$. Thus, the exponential IOWAD operator is: $\log_\gamma\left(\sum_{j=1}^{n} w_j \gamma^{b_j}\right)$, if $\gamma \neq 1$, and the IOWAD if $\gamma = 1$.

The radical IOWAD is found if $\gamma > 0$, $\gamma \neq 1$, and the generating function is $g(t) = \gamma^{1/t}$. Thus, the radical IOWAD operator is:

$$f(\langle u_1, x_1, y_1\rangle,\ldots,\langle u_n, x_n, y_n\rangle) = \left(\log_\gamma\left(\sum_{j=1}^{n} w_j \gamma^{1/b_j}\right)\right)^{-1} \quad (28)$$

Another interesting particular type of Quasi-IOWAD operator is the Quasi-weighted averaging distance (Quasi-WAD). It is found when $u_i > u_{i+1}$, for all i. It can be defined as follows.

Definition 8. *A Quasi-WAD operator is a mapping f: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:*

$$f(\langle x_1, y_1\rangle,\ldots,\langle x_n, y_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\big(b_{(j)}\big)\right) \quad (29)$$

where b_j is the $|x_i - y_i|$ value of the QWAD tuple $\langle x_i, y_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, $|x_i - y_i|$ is the argument variable represented in the form of individual distances, and g is the strictly continuous monotonic function.

Note that if $w_j = 1/n$, for all i., then, the Quasi-WAD becomes the Quasi-normalized averaging distance (Quasi-NAD).

Note that all the properties studied in the IMOWAD and in the Quasi-IOWAD can also be studied with the Quasi-WAD operator.

6. Numerical Example

The IMOWAD operator can be applied in a wide range of problems including statistics, economics and engineering. In the following, we are going to develop a simple illustrative example in order to see the results obtained in the aggregation by using different types of IMOWAD operators. We develop an application in a decision making problem concerning the selection of investments where an enterprise is looking for the best strategy according to his interests.

Assume that an enterprise that operates in Europe and North America wants to invest some money the next year. In order to do so, the board of directors, after careful analysis with the group of experts of the company, has established five possible investments S_i that the enterprise could develop in the future.

- A_1: Invest in the Asian market.
- A_2: Invest in the South American market.

- A_3: Invest in the African market.
- A_4: Do not develop any investment.

After careful review of the information, the experts have given the following general information. They have summarized the information of the strategies in five main characteristics C_i with the following results.

- C_1: Risk of the investment.
- C_2: Benefits in the short term.
- C_3: Benefits in the long term.
- C_4: Difficulty of the investment.
- C_5: Other aspects.

Note that the results are valuations between 0 and 1.

Table 1. Available investments

	C_1	C_2	C_3	C_4	C_5
A_1	0.7	0.9	0.8	0.7	0.3
A_2	0.6	0.8	0.7	0.5	0.8
A_3	0.5	0.6	0.8	0.4	0.9
A_4	0.8	0.5	0.6	0.8	0.6

According to the objectives and policies of the enterprise, the experts have established the ideal investment for the company independently of the investments available. They have established the following valuations for it.

Table 2. Characteristics of the ideal investment

	C_1	C_2	C_3	C_4	C_5
I	0.8	0.9	1	0.9	0.9

In order to aggregate the information, the group of experts calculates the attitudinal character of the enterprise. Due to the fact that the attitudinal character depends upon the opinion of several members of the board of directors, it is very complex. Therefore, they need to use order inducing variables in the reordering process. The results are shown in Table 3.

Table 3. Order inducing variables

	C_1	C_2	C_3	C_4	C_5
A_1	12	16	20	24	8
A_2	22	18	20	26	28
A_3	14	20	15	18	17
A_4	26	21	19	15	13

With this information, it is possible to develop different methods for selecting an investment. In this example, we consider the NHD, the NED, the WHD, the WED, the OWAD, the IOWAD, the AIOWAD and the EIOWAD operator. Note that the weighting vector used is: W = (0.1, 0.2, 0.2, 0.2, 0.3). The aggregated results obtained by using the previous particular cases of IMOWAD operators are shown in Tables 4 and 5.

Table 4. Aggregated results 1

	NHD	WHD	OWAD	IOWAD
A_1	0.22	0.27	0.16	0.26
A_2	0.22	0.21	0.19	0.22
A_3	0.26	0.23	0.21	0.26
A_4	0.24	0.27	0.2	0.27

Table 5. Aggregated results 2

	AIOWAD	EIOWAD	Median	Olympic
A_1	0.18	0.349	0.2	0.1
A_2	0.22	0.249	0.2	0.3
A_3	0.26	0.306	0.3	0.233
A_4	0.21	0.305	0.3	0.3

As we can see, depending on the distance aggregation operator used, the optimal choice is different. Note that the lowest value in each method is the optimal result.

If we establish an ordering of the investments, a typical situation if we want to consider more than one alternative, we will get the following orders shown in Table 6. Note that the first alternative in each ordering is the optimal choice.

Table 6. Ordering of the investments

	Ordering
NHD	$A_3 \succ A_4 \succ A_1 = A_2$
WHD	$A_1 = A_4 \succ A_3 \succ A_2$
OWAD	$A_3 \succ A_4 \succ A_2 \succ A_1$
IOWAD	$A_4 \succ A_1 \succ A_3 \succ A_2$
AIOWAD	$A_3 \succ A_2 \succ A_4 \succ A_1$
EIOWAD	$A_1 \succ A_3 \succ A_4 \succ A_2$
Median-IOWAD	$A_3 = A_4 \succ A_1 = A_2$
Olympic-IOWAD	$A_2 = A_4 \succ A_3 \succ A_1$

As we can see, depending on the particular type of IMOWAD operator used, the results may lead to different decisions.

7. Conclusions

We have presented the IMOWAD operator. It is a distance measure that uses the IOWA operator in the Minkowski distance. The main advantage of this operator is that it generalizes a wide range of distances such as the NMD, the WMD, the MOWAD, the IOWAD and the EIOWAD operator. We have studied some of its main properties.

We have further generalized the IMOWAD operator by using quasi-arithmetic means. We have called it the Quasi-IOWAD operator. We have also studied the applicability of the IMOWAD and the Quasi-IOWAD operator. We have developed an application of the new approach in a decision making problem about selection of investments. We have seen that the main advantage of using the IMOWAD and the Quasi-IOWAD is that it gives a more complete view of the decision problem.

In future research, we expect to develop further extensions of the IMOWAD and the Quasi-IOWAD operator by adding new characteristics in the problem, following the research developed in Ref. 8, and applying them to other fields.

Acknowledgements

We would like to thank the anonymous referees for valuable comments that have improved the quality of the paper. Support from the Spanish Ministry of Science and Innovation under project JC2009-00189 and the University of Barcelona under project MC238206, is gratefully acknowledged.

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