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A Deterministic Algorithm for Min-max and Max-min Linear Fractional Programming Problems*

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Abstract

In this paper, a deterministic global optimization algorithm is proposed for solving min-max and max-min linear fractional programming problem (P) which have broad applications in engineering, management science, nonlinear system, economics and so on. By utilizing equivalent problem (Q) of the (P) and two-phase linear relaxation technique, the relaxation linear programming (RLP) about the (P) is established. The proposed algorithm is convergent to the global minimum of (P) through the successive refinement of the feasible region and solutions of a series of RLP. And finally the numerical examples are given to illustrate the feasibility of the presented algorithm.

Keywords min-max and max-min linear fractional programming, global optimization, two-phase linearization relaxation, branch and bound technique.

1. Introduction

Fractional programming is one of the most successful fields in nonlinear optimization. The optimization of min-max and max-min several linear fractional functions is a special class of optimization between fractional programming. It has attracted the interest of practitioners and researchers for at least 30 years (Refs. 1-8). During the past 10 years, interest in these problems has been especially intense. In part, this is because since its initial development it has spawn a wide variety of application, specially multi-stage stochastic shipping, cluster analysis and multi-objective bond portfolio and so on. Another reason for the strong interest in minmax linear fractional programming problem is that from a research point of view these

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problems pose significant theoretical and computational challenges. This is mainly because these problems are global optimization problem, i.e. they are known to generally possess multiple local optima that are not globally optima. Purpose of this paper is to develop an efficient and reliable algorithm for min-max and max-min several linear fractional functions over a polytope:

$$(\mathrm{P}): \begin{cases} \min \max\{\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\} \\ s.t.\ Ax \le b, \end{cases}$$

and

$$(\mathrm{P1}): \begin{cases} \max \min\{\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\} \\ s.t.\ Ax \le b, \end{cases}$$

where $A \in R^{m\times n}$, $b \in R^m$, $f_j(x)$ and $g_j(x), j = 1,\ldots, p,$ are all linear affine functions. Since problem (P1) can be transformed into equivalent problem (P), therefore, in the following we will only consider solving problem (P).

The above min-max and max-min linear fractional programming problem usually arise in engineering, management science, nonlinear system, economics and applied mathematics which have been reviewed in Refs. 1 and 9. For examples, in engineering and economics it is usually used to minimize a ratio of functions between a given period of time and a utilized resource in order to measure the efficiency or productivity of a system, in these types of problems the objective function is usually given as a min-max or max-min linear fractional functions (see Stancu-Minasian Ref. 10).

Min-max linear fractional programming problems arise in the design of electronic circuits, and also appear in the formulation of discrete and continuous rational approximation problems with respect to the Chebyshev norm [11], in continuous rational games [12], in multiobjective programming [13], in engineering design as well as in some portfolio selection problems discussed by Bajona-xandri and Martinezlegaz [14]. A multi-facility location-queueing problem giving rise to problem (P) is introduced in Ref.15. Financial planning with multiple fractional goals is discussed in Ref.16. An application in computational geometry leading to a problem (P) can be found in Ref.17. In case of infinitely rather than finitely many fractional problem (P) is related to fractional semi-infinite programming [18]. Many applications in engineering give rise to such a problem (P) when a lower bound for the smallest eigenvalue of an elliptic differential operator is to be determined [19].

To our knowledge, many algorithms is proposed by utilizing duality theory for obtaining local optimum of problem (P). A rich dual theory in case of min-max several linear fractional functions exists and is reviewed in Ref. 9. Many approaches essentially lead to the same dual

$$\min\{\sup_{x\in C} \frac{v^T F(x) - u^T H(x)}{v^T G(x)} : u \ge 0, v \ge 0, v \ne 0\}$$

where $F = \{f_1,\ldots, f_p\}, G = (g_1,\ldots, g_p)$. A rather direct approach to such a dual theory for problem (P) is given in Ref. 20. In Ref. 21, the authors modified the duality concepts proposed in Ref. 22. In Ref. 23, the authors use Clarke's generalized gradient [24] to introduce three types of dual programs and derive corresponding duality relations. An extensively recent treatment of duality for (P) is given in Ref. 25.

Except for dual methods, large number of algorithms proposed by finding an optimal solution of the parametric program

$$\pi(q) = \max_{x\in S}(\min_{1\le i\le p}[f_i(x) - qg_i(x)]),$$

which is the unique root $\bar{q}$ of $\pi(q) = 0,$ are the most popular algorithms for solving problem (P). In Ref. 26, various modifications of the original algorithm have been proposed and tested. In Ref. 27, Gugat presented a version which is always super-linearly convergent. In Ref. 28, the authors extended the interior-point algorithm to solve the max-min linear fractional programming problems. Meanwhile several interior-point algorithms have been proposed for the (P), which is convergent in polynomial time. In Ref. 29, an unified algorithm based on monotonic optimization theory [30] is proposed for generalized linear fractional programming problem (P), which combines cutting plane and branch-and-bound techniques.

Recently, in Ref. 31, the authors show that a minimax fractional programming problem is equivalent to a minimax nonfractional parametric problem for a given parameter in complex space, and the necessary and sufficient optimality conditions of nondifferentiable minimax fractional programming problem with complex variables under generalized convexities is established. In Ref. 32, an improved SQP method is proposed for solving minimax problems, and a new method with small computational cost is proposed to avoid the Maratos effect. In addition, its global and superlinear convergence are obtained under some suitable conditions. In Ref.33, Dinkelbach's global optimization approach for finding the global maximum of the fractional programming problem is discussed. Using the lagrangian function definition for this type of problem, the Kuhn–Tucker saddle point and stationary-point problems are established. In addition, via the concepts of Mond–Weir type duality and Schaible type duality, a general dual problem is formulated and some weak, strong and converse duality theorems are proven. In Ref. 34, optimality conditions are proved for a class of generalized fractional minimax programming problems involving $B(\rho,r)$ -invexity functions. Subsequently, these optimality conditions are utilized as a basis for constructing various duality models for this type of fractional programming problems and proving appropriate duality theorems.

In this paper, we will present a deterministic algorithm using branch and bound technique for globally solving problem (P). By utilizing equivalent problem and two-phase linear relaxation technique, the proposed algorithm is convergent to the global minimum of the (P) through the successive refinement of the linear relaxation of feasible region of the objective function and solutions of a series of (RLP). And finally numerical examples show the feasibility of the proposed algorithm.

2. Equivalent Problem and Linearization Technique

For solving problem (P), we first solve the following $2n$ linear programming problems:

$$\underline{x}_i=\begin{cases}\min\ x_i\\ s.t.\ \ Ax\le b,\end{cases}$$

and

$$\overline{x}_i=\begin{cases}\max\ x_i\\ s.t.\ \ Ax\le b,\end{cases}$$

where $i=1,2,\dots,n.$ We can easily obtain the initial lower bound and upper bound of each variable. Then we derive an initial rectangle X^0 denoted by: $X^0=\{x\,|\,\underline{x}_i\le x_i\le \overline{x}_i, i=1,2,\dots,n\}$.

Let

$$h_j(x)=\frac{f_j(x)}{g_j(x)},\ j=1,2,\dots,p,$$

we introduce the equivalent problem (Q) of the (P) as follows.

$$(\mathrm{Q}):\begin{cases}\min\ t\\ s.t.\ \ h_j(x)-t\le 0,\ j=1,2,\dots,p,\\ \quad\ \ Ax\le b,\\ \quad\ \ x\in X^0.\end{cases}$$

Theorem 1. *Problem (P) and (Q) have the same global optimal solutions and optimal value.*

In the following, we only consider solving problem (Q), in this paper principal construction in the development of a solution procedure for solving problem (Q) is construction of a linear relaxation programming for obtaining the lower bounds of the optimal value for this problem. The concept of linear bounding functions is introduced. For this considered problem (Q), we only need to construct linear lower bounding function of $h_j(x)$ in each constraint function. In this section the developed method uses a convenient linearization technique to derive the linear lower bounding function of every $h_j(x), j=1,2,\dots,p.$

Assumption 1. $f_j(x)>0$ *and* $g_j(x)>0,$ *for each* $j=1,2,\dots,p.$

Certainly, if the following situation $f_j(x)<0$ and $g_j(x)>0,$ or $f_j(x)>0$ and $g_j(x)<0,$ hold, then the $h_j(x)$ can be expressed as $-\frac{-f_j(x)}{g_j(x)}$ or $-\frac{f_j(x)}{-g_j(x)}$ which can be solved by the similar method proposed in this paper.

In the following, for each $h_j(x)$ in constraint function, we need to construct a linear lower bounding function. By assumption 1, we can let $\eta_j=\ln(f_j(x))$ and $\xi_j=\ln(g_j(x))$, where $j=1,2,\dots,p,$ then we have $h_j(x)=\exp(\eta_j-\xi_j).$ Here we adopt two-phase linear relaxation method (Ref. 5). In the first-phase a linear lower bounding function about the variables η_j and ξ_j is derived. Then in the second-phase the linear lower bounding function about the primal variable x is constructed ultimately.

2.1. *First-phase relaxation*

By the convexity of the $\exp(Z)$, we can give a linear lower bound function of the $\exp(Z)$ over the interval $[Z^l,Z^u]$ as follows:

$$L(\exp(Z))=B(1+Z-\ln B),$$

where

$$B=\frac{\exp(Z^u)-\exp(Z^l)}{Z^u-Z^l},$$

which is called tangential approximation at the point $\overline{Z}=\ln B.$

Additionally we can give a linear upper bound function of $\exp(Z)$ over the interval $[Z^l,Z^u]$ as follows:

$$U(\exp(Z))=B(Z-Z^l)+\exp(Z^l).$$

Based on the above discussion, for each function $h_j(x)$ in constraints, we can construct the corresponding first-phase lower bound function. According to the considered rectangle about the primal variable x the bound about η_j and ξ_j can also be obtained. Denote by η_j^u , η_j^l the upper bound and lower bound of η_j , and by ξ_j^u , ξ_j^l the upper bound and lower bound of ξ_j on the considered rectangle in the algorithm. Denote by T_j^l, T_j^u the lower bound and the upper bound of $\eta_j - \xi_j$ which can be derived easily.

Then we have

$$B_j(1+\eta_j-\xi_j-\ln B_j)\le \exp(\eta_j-\xi_j)$$
$$\le B_j(\eta_j-\xi_j-T_j^l)+\exp(T_j^l),$$

where

$$B_j=\frac{\exp(T_j^u)-\exp(T_j^l)}{T_j^u-T_j^l},$$

Let

$$Lh_j(\eta_j-\xi_j)=B_j(1+\eta_j-\xi_j-\ln B_j),$$

then the first-phase lower bound function of $h_j(x)$ about variable x , for some $j, j=1,\dots,p,$ is given as follow:

$$Lh_j(\eta_j-\xi_j)=Lh_j(x)$$
$$=B_j(1+\ln(f_j(x))-\ln(g_j(x))-\ln B_j). \quad (1)$$

2.2. *Second-phase relaxation*

Since the $\ln(Z)$ is a concave function, in the similar method as the first-phase relaxation we can obtain a linear lower bound function of the $\ln(Z)$ over the interval $[Z^l, Z^u]$ as follows:

$$L(\ln(Z))=C(Z-Z^l)+\ln Z^l$$

where

$$C=\frac{\ln Z^u-\ln Z^l}{Z^u-Z^l},$$

which is called tangential approximation at the point $\bar{Z}=\frac{Z^u-Z^l}{\ln Z^u-\ln Z^l}$.

Additionally we can give a linear upper bound function of $\ln(Z)$ over the interval $[Z^l, Z^u]$ as follows:

$$U(\ln(Z))=CZ-1-\ln C.$$

And $L(\ln(Z))$, $U(\ln(Z))$ and $\ln(Z)$ satisfies the following inequality:

$$C(Z-Z^l)+\ln Z^l\le\ln(Z)\le CZ-1-\ln C.$$

Therefore, by the above discussion we can obtain the following inequality:

$$C_j(f_j(x)-\exp(\eta_j^l))+\eta_j^l\le\ln(f_j(x))$$
$$\le C_jf_j(x)-1-\ln C_j, \quad (2)$$

$$D_j(g_j(x)-\exp(\xi_j^l))+\xi_j^l\le\ln(g_j(x))$$
$$\le D_jg_j(x)-1-\ln D_j, \quad (3)$$

where

$$C_j=\frac{\eta_j^u-\eta_j^l}{\exp(\eta_j^u)-\exp(\eta_j^l)},$$
$$D_j=\frac{\xi_j^u-\xi_j^l}{\exp(\xi_j^u)-\exp(\xi_j^l)}.$$

by the above inequality (1) (2) and (3), then finally we derive the linear lower bounding function $L_j(x)$ of $h_j(x)$ for some $j=1,2,\dots,p,$ which underestimates the value of the considered function $h_j(x)$ as follows:

$$L_j(x)=B_j\{2+C_j[f_j(x)-\exp(\eta_j^l)]+\eta_j^l$$
$$-D_j(g_j(x))+\ln D_j-\ln B_j\}.$$

Obviously, $L_j(x)\le h_j(x),$ for $\forall x\in X^k\subseteq X^0$.

2.3. *Relaxation linear programming*

Therefore, for $\forall X^k\subseteq X^0$, we can construct the relaxation linear programming (RLP) of the (Q) in X^k as follows:

$$(\text{RLP}):\begin{cases}\min & t\\ s.t. & L_j(x)-t\le 0, j=1,2,\dots,p,\\ & Ax\le b,\quad x\in X^k.\end{cases}$$

Theorem 2. *Let* $\omega_j=T_j^u-T_j^l, v_j=\frac{\eta_j^u}{\eta_j^l}, u_j=\frac{\xi_j^u}{\xi_j^l},$ *then the error* $\Theta_j=h_j(x)-L_j(x)\to 0$ *as* $\omega_j\to 0,$ $j=1,2,\dots,p$.

Proof. Similarly as the proof of Lemma 1 in Ref. 5, let $\Theta_j=[h_j(x)-Lh_j(x)]+[Lh_j(x)-L_j(x)]=\Theta_{j1}+\Theta_{j2}$, then, first, we consider the difference Θ_{j1}. It follows

$$\Theta_{j1}=h_j(x)-Lh_j(x)$$
$$=\exp(\eta_j-\xi_j)-B_j(1+\eta_j-\xi_j-\ln B_j).$$

Since Θ_{j1} is a convex function about $(\eta_j-\xi_j)$, for any $(\eta_j-\xi_j)\in[T_j^l, T_j^u]$ defined in former. Then it follows that Θ_{j1} can obtain the maximum $\Theta_{j1}^{\max}$ at the points T_j^l or T_j^u . Let

$$z_j=\frac{\exp(\omega_j)-1}{\omega_j},$$

then through computing we can derive the following form:

$$\Theta_{j1}^{\max} = \Theta_{j1}(T_j^l) = \Theta_{j1}(T_j^u) = \exp(T_j^l)(1 - z_j + z_j \ln z_j).$$

Since $z_j \to 1$, as $\omega_j \to 0$. So it is obvious that $\Theta_{j1}^{\max} \to 0$ as $\omega_j \to 0$.

Secondly, similarly we consider the difference Θ_{j2}, it follows that

$$\begin{aligned}\Theta_{j2} &= Lh_j(x) - L_j(x) \\ &= B_j\{\ln(f_j(x)) - C_j[f_j(x) - \exp(\eta_j^l)] - \eta_j^l\} \\ &\quad + B_j\{D_j g_j(x) - 1 - \ln D_j - \ln(g_j(x))\}.\end{aligned}$$

Let

$$\Theta_{j2.1} = \ln(f_j(x)) - C_j(f_j(x) - \exp(\eta_j^l)) - \eta_j^l,$$

$$\Theta_{j2.2} = D_j g_j(x) - 1 - \ln(D_j) - \ln(g_j(x)),$$

Then

$$\Theta_{j2} = B_j \Theta_{j2.1} + B_j \Theta_{j2.2} \; . \qquad (4)$$

Since $\Theta_{j2.1}$ is a concave function about $f_j(x)$, we can know $\Theta_{j2.1}$ can attain the maximum $\Theta_{j2.1}^{\max}$ at the point $f_j(x) = \ln(1/C_j)$. Then through computing we derive:

$$\Theta_{j2.1}^{\max} = \frac{\ln v_j}{v_j - 1} - 1 - \ln\frac{\ln v_j}{v_j - 1}.$$

Also, since $\Theta_{j2.2}$ is a convex function, it follows that it can attain the maximum $\Theta_{j2.2}^{\max}$ at the point ξ_j^u or ξ_j^l. Then through computing we derive

$$\Theta_{j2.2}^{\max} = \frac{\ln u_j}{u_j - 1} - 1 - \ln\frac{\ln u_j}{u_j - 1}.$$

Since $v_j \to 1$ and $u_j \to 1$ as $\omega_j \to 0$, then we have $\Theta_{j2.1}^{\max} \to 0$ and $\Theta_{j2.2}^{\max} \to 0$ as $\omega_j \to 0$. Therefore from (4) we can follow that $\Theta_{j2}^{\max} \to 0$ as $\omega_j \to 0$. By the above discussion it is obvious that the conclusion is followed. □

The above theorem ensures each $L_j(x)$ will approximate the corresponding function $h_j(x)$ as $\omega_j \to 0, j = 1, \ldots, p$. Obviously, we have the optimal value of the RLP and (Q) satisfy $v(\text{RLP}) \le v(\text{Q})$, i.e. the RLP provides the lower bound for the optimal value of the equivalent problem (Q). Based on the above construction method of the linear relaxation, for $\forall X^k \subseteq X^0$, problem RLP($X^k$) provides a valid lower bound for the optimal value of problem Q(X^k).

3. Algorithm and Its convergence

In this section, a deterministic global optimization algorithm is developed to solve the (Q) based on the former linear relaxation technique. This algorithm needs to solve a sequence of relaxation linear programming over partitioned subsets of X^0 in order to find a global optimum solution. Furthermore, in order to ensure convergence to a global optimum, some bound tightening strategies can be applied to enhance the solution procedure.

The branch and bound approach is based on partitioning the set X^0 into sub-hyper-rectangles, each concerned with a node of the branch and bound tree, and each node is associated with a relaxation linear sub-problem in each sub-hyper-rectangle. Hence, at any stage k of the algorithm, suppose that we have a collection of active nodes denoted by Ω_k, say, each associated with a hyper-rectangle $X \subseteq X^0$, $X \in \Omega_k$. For each such node X, we will have computed a lower bound of the optimal value of the (Q) via the solution $LB(X)$ of the RLP, so that the lower bound of optimal value of the (Q) on the whole initial box region X^0 at stage k is given by $LB_k = \min\{LB(X), \forall X \in \Omega_k\}$. Whenever the lower bounding solution for any node sub-problem, i.e., the solution of the relaxation linear programming RLP turns out to be feasible to the (Q), we update the upper bound of incumbent solution UB if necessary. Then, the active nodes collection Ω_k will satisfy $LB(X) < UB, \forall X \in \Omega_k$, for each stage k. We now select an active node to partition its associated hyper-rectangle into two sub-hyper-rectangles as described below, computing the lower bounds for each new node as before. Upon fathoming any non-improving nodes, we obtain a collection of active nodes for the next stage, and this process is repeated until convergence is obtained.

Branching rule:

The critical element in guaranteeing convergence to a global minimum is the choice of a suitable partitioning strategy. In our paper we choose a simple and standard bisection rule. This method is sufficient to ensure convergence since it drives all the intervals to zero for the variables that are associated with the term that yields the greatest discrepancy in the employed approximation along any infinite branch of the branch and bound tree. This branching rule is as follows.

Consider any node sub-problem identified by the hyper-rectangle $X' = [\underline{x}, \overline{x}] \subseteq X^0$, and the selection of the branching variable x^q and the partitioning of X' is then done using the following rule.

let

$$q = \arg\max\{\overline{x}_i - \underline{x}_i : i = 1,\ldots,n\} ,$$

and partition X' by bisection the interval $[\underline{x}_q, \overline{x}_q]$ into the subintervals $[\underline{x}_q, (\underline{x}_q + \overline{x}_q)/2]$ and $[(\underline{x}_q + \overline{x}_q)/2, \overline{x}_q]$.

The basic steps of the proposed global optimization algorithm are summarized as follows. Let $LB(X^k)$ refer to the optimal objective function value of (RLP) on the sub-hyper-rectangles X^k and $x^k = x(X^k)$ refer to an element of corresponding argmin.

Step 1: (Initialization)

Initialize the convergence tolerance ε; the iteration counter $k := 0$; the set of active node $\Omega_0 = X^0$; the upper bound $UB = \infty$; the set of feasible points $F := \varnothing$.

Solve the RLP(X) for $X = X^0$, obtaining $LB_0 := LB(X^0)$ and $x^0 = x(X^0)$. If x^0 is feasible to the (P), update F and UB, if necessary. If $UB - LB_0 \le \varepsilon$, then stop with x^0 as the prescribed solution to problem (P). Otherwise, proceed to step 2.

Step 2: (Updating upper bound)

Let

$$UB = \min_{x\in F} \max\left\{\frac{f_1(x)}{g_1(x)},\ldots,\frac{f_p(x)}{g_p(x)}\right\},$$

If $F := \varnothing$, then the known best feasible solution is

$$\tilde{x} = \arg\min_{x\in F} \max\left\{\frac{f_1(x)}{g_1(x)},\ldots,\frac{f_p(x)}{g_p(x)}\right\}.$$

Step 3: (Partition)

Partition X^k to get two new sub-hyper-rectangles according to the above branching rule. Call the set of new partition rectangles as $\overline{X}^k$. For each $X \in \overline{X}^k$, calculate the lower bound $LB(X)$ and $x(X)$ by solving RLP(X).

If $LB(X) > UB,$ then let $\overline{X}^k := \overline{X}^k \setminus X$.

Step 4: (Updating lower bound)

If $x(X)$ satisfy $LB(X) \le UB$ and $x(X)$ is feasible to problem (P), then update UB, F and $\tilde{x}$, if necessary, and let

$$\Omega_k = (\Omega_k \setminus X) \bigcup \overline{X}^k,$$

update lower bound

$$LB_k = \inf_{X\in\Omega_k} LB(X).$$

Step 5: (Fathoming)

Let

$$\Omega_{k+1} = \Omega_k \setminus \{X : UB - LB(X) \le \varepsilon, X \in \Omega_k\}.$$

If $\Omega_{k+1} = \varnothing$, then algorithm stop, UB is the global optimal value for the (P), $\tilde{x}$ is a global optimization solution of problem (P);

Otherwise, let $k := k+1$, select X^k such that $X^k = \arg\min_{X\in\Omega_k} LB(X)$, $x^k := x(X^k)$, return to step 2.

Theorem 3. *The above algorithm either terminates finitely with the solution being optimal to the (P), or generates an infinite sequence of iteration such that along any infinite branch of the branch-and-bound tree, and accumulation point of the sequence $\{LB_k\}$ will be the global minimum of the (P).*

Proof. In Ref. 35, Horst and Tuy point out that a sufficient condition for convergence of the algorithm is that the bounding operation must be consistent and the selection operation bound improving.

Let LB_k is a computed lower bound in stage k and UB is the best upper bound at iteration k not necessarily occurring inside the same sub-rectangle with LB_k. Since the employed subdivision process is the bisection, the process is exhaustive. Consequently, from Theorem 2 the following formulation holds:

$$\lim_{k\to\infty}(UB - LB_k) = 0,$$

and then it means that the employed bounding operation is consistent.

Obviously, since the partition element where the actual lower bound is attained is selected for further partition in the immediately following iteration, the employed selection operation is called for bound improving.

In summary, the proposed algorithm are satisfied with that the bounding operation is consistent and that the selection operation is bound improving, therefore according to Theorem IV.3. in Horst and Tuy [35] the employed algorithm is convergent to the global minimum of problem (P). □

4. Numerical examples

To verify the feasibility of the proposed algorithm, compute the following several numerical examples using C++ code in Pentium IV.

Example 1 [29] .

$$\begin{cases} \max\min\left\{\dfrac{37x_1+73x_2+13}{13x_1+13x_2+13},\dfrac{63x_1-18x_2+39}{13x_1+26x_2+13}\right\} \\ s.t. \quad 5x_1-3x_2=3, \\ \quad 1.5\le x_1\le 3, \\ \quad 1.5\le x_2\le 4. \end{cases}$$

Set $\varepsilon=5\times10^{-8}$, numerical results is given as follows: number of iteration $s=6$, the maximal number of active nodes necessary $n=7$, execution time in seconds $t=0s$, obtain the optimal value $V^*=1.49072061$, the optimal solution $x_1^*=1.5$, $x_2^*=1.5$.

Example 2 [29] **.**

$$\begin{cases} \min\max\left\{\dfrac{3x_1+x_2-2x_3+0.8}{2x_1-x_2+x_3},\dfrac{4x_1-2x_2+x_3}{7x_1+3x_2-x_3}\right\} \\ s.t. \quad x_1+x_2-x_3\le 1, \\ \quad -x_1+x_2-x_3\le -1, \\ \quad 12x_1+5x_2+12x_3\le 34.8, \\ \quad 12x_1+12x_2+7x_3\le 29.1, \\ \quad -6x_1+x_2+x_3\le -4.1, \\ \quad 1.0\le x_1\le 1.1, \\ \quad 0.55\le x_2\le 0.65, \\ \quad 1.35\le x_3\le 1.45. \end{cases}$$

Set $\varepsilon=5\times10^{-8}$, numerical results is given as follows: number of iteration $s=6$, the maximal number of active nodes necessary $n=5$, execution time in seconds $t=0s$, obtain the optimal value $V^*=0.572810738$, the optimal solution $x_1^*=1.015678086$, x_2^*=0.590676676, $x_3^*=1.403391837$.

Example 3.

$$\begin{cases} \min\max\left\{\dfrac{2x_1+2x_2-x_3+0.9}{x_1-x_2+x_3},\dfrac{3x_1-x_2+x_3}{8x_1+4x_2-x_3}\right\} \\ s.t. \quad x_1+x_2-x_3\le 1, \\ \quad -x_1+x_2-x_3\le -1, \\ \quad 12x_1+5x_2+12x_3\le 34.8, \\ \quad 12x_1+12x_2+7x_3\le 29.1, \\ \quad -6x_1+x_2+x_3\le -4.1. \\ \quad 1.0\le x_1\le 1.2, \\ \quad 0.55\le x_2\le 0.65, \\ \quad 1.35\le x_3\le 1.45, \end{cases}$$

Set $\varepsilon=5\times10^{-8}$, numerical results is given as follows: number of iteration $s=8$, the maximal number of active nodes necessary $n=8$, execution time in seconds $t=0s$, obtain the optimal value $V^*=1.346854863$, the optimal solution is $x_1^*=1.016666667$, $x_2^*=0.55$, $x_3^*=1.45$.

Example 4.

$$\begin{cases} \min\max\left\{\begin{matrix}\dfrac{3x_1+x_2-2x_3+0.8}{2x_1-x_2+x_3},\dfrac{4x_1-2x_2+x_3}{7x_1+3x_2-x_3}, \\ \dfrac{3x_1+2x_2-x_3+1.9}{x_1-x_2+x_3},\dfrac{4x_1-x_2+x_3}{8x_1+4x_2-x_3}\end{matrix}\right\} \\ s.t. \quad x_1+x_2-x_3\le 1, \\ \quad -x_1+x_2-x_3\le -1, \\ \quad 12x_1+5x_2+12x_3\le 34.8, \\ \quad 12x_1+12x_2+7x_3\le 29.1, \\ \quad -6x_1+x_2+x_3\le -4.1, \\ \quad 1.0\le x_1\le 1.2, \\ \quad 0.50\le x_2\le 0.65,\ 1.35\le x_3\le 1.45. \end{cases}$$

Set $\varepsilon=5\times10^{-8}$, the results is given as follows: number of iteration $s=7$, the maximal number of active nodes necessary $n=8$, execution time in seconds $t=0s$, obtain the optimal value $V^*=2.284427051$, the optimal solution is $x_1^*=1.008333333$, $x_2^*=0.5$, $x_3^*=1.45$.

Numerical result show that our algorithm can globally solve min-max and max-min linear fractional programming problem (P) and (P1) on a microcomputer.

5. Conclusion

In this paper, by utilizing equivalent transformation, two-phase linear relaxation method and branch-and-bound technique, we present a deterministic algorithm for solving min-max and max-min linear fractional programming problem (P) and (P1) which have broad applications in engineering, management science, nonlinear system, economics and so on. The proposed algorithm is convergent to the global minimum of (P) through the successive refinement of the feasible region and solutions of a series of RLP. And finally several numerical examples are given to illustrate the feasibility and efficiency of the presented algorithm.

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