Existence and Uniqueness of Almost Periodic Solution for A Class of Nonlinear Delay Integro-differential Equation

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Keywords: Delay nonlinear system; Almost periodic solution; Existence; Lyapunov functional; Globally uniformly asymptotically stable.

Abstract: In this paper, we consider a general delay nonlinear system. The existence of almost periodic solution of the system is discussed under some relatively weaker conditions by means of constructing a class of Lyapunov functional and asymptotic almost periodic functions. Finally, some examples are given to support our theoretical analyses.

Introduction

By using the exponential dichotomy and fixed point theorem, Chen [1] discussed the existence problem of T -periodic solutions for the periodic system

$$x'(t) = A(t, x(t))x(t) + \int_{-\infty}^{t} C(t,s)x(s)ds + f(t, x(t-\tau)) + b(t) \qquad (1.1)$$

Kato and Imai [2] considered the periodic and almost periodic solutions for the nonlinear system

$$x'(t) = f(t,x) + e(t), \qquad (1.2)$$

and obtained the existence and uniqueness of the solutions. Under the condition that the system

$$x'(t) = f(t,x) + \int_{-\infty}^{0} g(t,s,x(t+s),x(t))ds + h(t,x_t), \qquad (1.3)$$

Has a bounded solution and time delay $|h(t,x)| \to 0$, Hamaya [3] established the existence of solutions.

For a nonlinear time-delay system, however, it is difficult to determine whether the system has a bounded solution. Furthermore, the condition $|h(t,x)| \to 0$ is also a strong restriction. In this paper, we consider the following system

$$x'(t) = f(t,x) + \int_{-\infty}^{t} C(t,s)x(s)ds + g(t,x_t) + e(t) \qquad (1.4)$$

Under some weaker conditions, the existence of almost periodic solutions of the system is established.

Throughout this paper, $x \in R^n, \|.\|$ denotes a fixed norm, $f(t,x)$ is about t to t, $g(t,\varphi)$ is about t to φ uniform almost periodic continuous function, $e(t)$ is an almost periodic continuous function of t, and $C(t,s)$ is an almost periodic continuous function of (t,s),that is, for any $\varepsilon > 0$ there exists $\tau > 0$ such that $|C(t+\tau, s+\tau) - C(t,s)| < \varepsilon$.

Existence of Almost Periodic Solutions

Firstly, we introduce some Lemmas which will be used in the following discussion

Lemma 1.1[3] The functional $[,]: R^n \times R^n \to R$ defined by the $[x,y]=\lim_{h\to 0}\frac{\|x+hy\|-\|x\|}{h}$ has the following properties

(1) $[x,y]=\inf_{h>0}\frac{\|x+hy\|-\|x\|}{h}$; (2) $\|[x,y]\|\le\|y\|$;

(3) $[x,y+z]\le[x,y]+[x,z]$; (4) $D^+\|u(t)\|=[u(t),u'(t)]$,

where $D^+\|u(t)\|$ denotes the upper-right derivative of $\|u(t)\|$

Lemma 1.2 [4] For almost periodic functions $p(t)$ and arbitrary $b\in R$

$\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t p(u+b)du=\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t p(u)du$ Furthermore, if $\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t p(u)du=-\gamma<0$, then for any $b\in R$ there exist positive constants α,β which are independent of b such that

$$\exp(\int_s^t p(u+b)du)=\beta\exp(-\alpha(t-s)),\quad (t>s).$$

Now, we give our results and the proof as follows

Theorem 2.1 Suppose the following conditions hold

(H2.1) There is an almost periodic continuous function $p(t)$ and a constant $\gamma>0$ satisfying

$$\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t p(u)du=-\gamma;$$

(H2.2) For any $(t,x),(t,y)\in R\times R^n$ and any almost periodic function $p(t)$ satisfying condition (H2.1), $[x-y, f(t,x)-f(t,y)+\int_{-\infty}^t C(t,s)x(s)ds-\int_{-\infty}^t C(t,s)y(s)ds\le p(t)\|x-y\|$;

(H2.3) $\int_{-\infty}^t\|C(t,s)\|ds<+\infty$,and for any $\varepsilon>0$,there exists $S>0$ such that $\int_{-\infty}^{-S}\|C(t,s)\|ds<\varepsilon$

(H2.4) $\lim_{n\to\infty}\frac{1}{n}\sup_{\|\varphi\|\le n}\|g(t,\varphi)\|=0$;

(H2.5) $f(t,0)=0$ and there exists a constant $L>0$ such that $\|e(t)+g(t,0)\|<L$ Then the system (1.4) has a unique almost periodic solution

Proof. Consider the following system
$$x'(t)=f(t,x)+\int_{-\infty}^t C(t,s)x(s)ds+e(t)\qquad(2.5)$$

Firstly, We will prove the system (2.5) has a bounded solution. Assuming that $x(t)$ is a solution of (2.5), then we have

$$\begin{aligned}D^+\|x(t)\|&=[x(t),x'(t)]\\&=[x(t), f(t,x)+\int_{-\infty}^t C(t,s)x(s)ds+e(t)]\\&\le[x(t), f(t,x)+\int_{-\infty}^t C(t,s)x(s)ds]+[x(t),e(t)]\\&\le p(t)\|x(t)\|+\|e(t)\|\le p(t)\|x(t)\|+L.\end{aligned}\qquad(2.6)$$

By solving the differential inequality, we get $\|x(t)\|\le\beta\|x(0)\|+\frac{\beta L}{\alpha}\quad(t>0).$ (2.7)

For the sequence $\{t_n\}, t_n \to +\infty (n \to \infty),$ according to (2.7) , we know that $\{x(t_n)\}$ is a bounded sequence. Using the properties of the almost periodic functions, as $n \to \infty$, we obtain $f(t+t_n, x) \to f(t,x);\ e(t+t_n) \to e(t);\ \int_{-\infty}^{t} C(t+t_n, s+t_n)x(s+t_n)ds \to \int_{-\infty}^{t} C(t,s)x(s)ds.$

Since $x(t+t_n)$ is the solution for the system

$$x'(t+t_n) = f(t+t_n, x(t+t_n)) + \int_{-\infty}^{t} C(t+t_n, s+t_n)x(s+t_n)ds + e(t+t_n), \qquad (2.8)$$

Letting $n \to \infty$ we have $x'(t+t_n) \to x^{*\prime}(t)$ and $x^{*\prime}(t) = f(t, x^*) + \int_{-\infty}^{t} C(t,s)x^*(s)ds + e(t),$ (2.9)
where $x^*(t)$ is a solution for the system (2.5). On the other hand, we have

$$\|x(t+t_n)\| \le \beta\|x(0)\| + \frac{\beta L}{\alpha} \qquad (t > -t_n) \qquad (2.10)$$

Letting $n \to \infty$ we can obtain $\|x^*(t)\| \le \beta\|x(0)\| + \frac{\beta L}{\alpha} \quad (t \in R),$ (2.11)

which means the system (2.5) has a bounded solution $x^*(t)(t \in R).$

Secondly, we will prove that the bounded solution of (2.5) is unique. Assuming that $y(t)$ is a solution of (2.5), then we have $D^+\|x^*(t) - y(t)\| \le p(t)\|x^*(t) - y(t)\|,$ furthermore,

$$\|x^*(t) - y(t)\| \le \|x^*(0) - y(0)\| \exp(\int_0^t p(\sigma)d\sigma) \le \beta\|x^*(0) - y(0)\| \exp(-\alpha t) \to 0 \quad (t \to \infty) \qquad (2.12)$$

Hence, the bounded solution is unique.

Thirdly, we will prove that the bounded sequence $\{x(t+t_n)\}$ is uniform convergent. For $D^+\|x(t+t_k) - x(t+t_m)\| \le p(t)\|x(t+t_k) - x(t+t_m)\| + \|e(t+t_k) - e(t+t_m)\|,$ We have

$$\|x(t+t_k) - x(t+t_m)\| \le \beta\|x(t_k) - x(t_m)\| + \frac{\beta}{\alpha}\|e(t+t_k) - e(t+t_m)\|.$$

Noticing that $\{x(t_n)\}$ is a bounded sequence, hence, for all $\varepsilon > 0$ there is a sufficiently large positive N_1 such that $\|x(t_k) - x(t_m)\| < \frac{\varepsilon}{3\beta}$ When $k, m \ge N_1$, Since $e(t)$ is an almost periodic continuous function, there exists a positive number N_2 such that $\|e(t+t_k) - e(t+t_m)\| < \frac{\alpha\varepsilon}{3\beta}$ when $k, m \ge N_2$.

Let $N_0 = \max\{N_1, N_2\}$, then we get $\|x(t+t_k) - x(t+t_m)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$ when $k, m \ge N_0$ which shows that $\{x(t+t_n)\}$ is uniform convergent.

Finally, we obtain that $x(t)$ is an asymptotically almost periodic solution of (2.5). From [5,10], there exists an almost periodic solution for the system (2.5). Since almost periodic functions are bounded and there is a unique bounded solution for the system (2.5), we can infer that there is a unique almost periodic solution for the system (2.5).

Defining a set $B = \{u(t) : u(t) \text{ is an almost periodic function}\},$ then B is a Banach space under the norm $\|u(t)\| = \sup_{t \in R}|u(t)|.$ For any $u(t) \in B,$

Consider the function $x_u(t) = f(t,x) + \int_{-\infty}^{t} C(t,s)x(s)ds + g(t,x_t) + e(t).$ (2.13) Define the operator T: $u(t) \to x_u(t).$ Under the conditions of the theorem, we need to prove that there exists a subset of B_0 such that $B_0 \to B_0$ is a completely continuous operator.

Denoting $B_n = \{u(t) : u(t) \in B \text{ and } \|u(t)\| \le n\},$ where n is a natural number. Then we can conclude that there exists a natural number N such that $T : B_N \to B_N.$ If not, there is $u_n \in B_n$ for any natural number n such that $\|Tu_n\| > n.$ By condition (H2.4), there is a natural number $N \ge \max\{4\beta\|u(0)\|, \frac{4\beta L}{\alpha}\}$ for $\varepsilon \le \frac{1}{4},$ when $n \ge N$ then we have $\frac{1}{n}\sup_{\|\varphi\|\le n}\|g(t,\varphi)\| \le \varepsilon \le \frac{1}{4}$ According to (2.7), (2.13), we can obtain $\frac{1}{n}\|Tu_n(t)\| \le \frac{\beta}{n}\|u_n(0)\| + \frac{\beta L}{n\alpha} + \frac{1}{n}\sup_{\|\varphi\|\le n}\|g(t,\varphi_n)\| \le \frac{3}{4}.$ This conflicts with $\|Tu_n\| > n.$ Hence, there is a natural number N such that $T : B_N \to B_N.$

Next, we will prove that the TB_N is a compact subset in B. Since $TB_N \subseteq B_N$, $\{Tu(t); u(t) \in B_N\}$ is uniformly bounded. Denote $A_1 = \sup_{t\in R}\{Tu(t); u(t) \in B_N\}$ Using the assumptions, $f(t,x)$ is the uniform almost periodic continuous function about t for x, then we can obtain that $f(t,u)$ is bounded when $\|u(t)\| \le N$ According to $\frac{1}{n}\sup_{\|\varphi\|\le n}\|g(t,\varphi)\| \to 0 (n \to \infty),$ we can infer that $\|g(t,\varphi)\|$ is bounded. Denoting $d_1 = \sup_{t\in R, \|x\|\le N}\|f(t,x)\|,\ d_2 = \sup_{t\in R}\int_{-\infty}^{t}\|C(t,s)\|ds,\ d_3 = \sup_{t\in R, \|x\|\le N}\|g(t,\varphi)\|,\ d_4 = \sup_{t\in R}\|e(t)\|,$

We have $\frac{dTu(t)}{dt} = \frac{dx_n}{dt} = f(t,x_n(t)) + \int_{-\infty}^{t} C(t,s)x_u(s)ds + g(t,x_{u_t}(t)) + e(t),$ furthermore, $\left\|\frac{dTu(t)}{dt}\right\| \le d_1 + d_2 N + d_3 + d_4.$ (2.15) Hence, $\{Tu(t); u(t) \in B_N\}$ is uniformly bounded and equicontinuous.

So by Ascoli theorem, TB_N is a compact subset of B_N

At last, we will prove T is continuous in B_N For any $\varepsilon > 0,$ by the condition (H2.1), there is sufficiently large $K > 0$ such that $\exp(\int_{t-K}^{t} p(s)ds) < \frac{\varepsilon}{4A_1}$ .Denoting $A_2 = \int_{t-K}^{t}\exp(\int_{s}^{t} p(\sigma)d\sigma)ds$, we know that f and g are continuous functions according to the assumptions. Hence, for any $u(t),\ v(t) \in B_N$, we have $\|g(t,u_t) - g(t,v_t)\| < \frac{\varepsilon}{2A_2},$ when $\|u_t - v_t\| < \delta(\varepsilon)$ .For

$D^+\|Tu(t) - Tv(t)\| \le p(t)\|Tu(t) - Tv(t)\| + \|g(t,u_t) - g(t,v_t)\|,$ we can obtain

$$\|Tu(t) - Tv(t)\| \le \|Tu(t-K) - Tv(t-K)\|\exp(\int_{t-K}^{t} p(s)ds$$

$$+\|g(t,u_t) - g(t,v_t)\|\int_{t-K}^{t}\exp(\int_{s}^{t} p(\sigma)d\sigma)ds \quad (2.16)$$

$$< 2A_1 \cdot \frac{\varepsilon}{4A_1} + \frac{\varepsilon}{2A_2} \cdot A_2 = \varepsilon,$$

Which means that T is continuous in B_N Therefore, T is completely continuous in B_N.

According to Schrader fixed point theorem, T has a fixed point in B_N,that is, there is an almost periodic solution for (1.4).

Obviously, this is the unique almost periodic solution.

Conclusions

In this paper, we have discussed the existence of almost periodic solution for a class of nonlinear delay integro-differential equations under some relatively weaker conditions by means of constructing a class of Lyapunov functional and asymptotic almost periodic functions. Some examples are given to support our theoretical analyses.

Acknowledgement

It is a project supported supported by Scientifics Research Foundation of the Education Department of Guangxi Zhuang Autonomous Region (no. **KY2015ZD103) .**The corresponding author is Zongyi Hou.

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