Estimation of unknown function of a class of integral inequalities

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Abstract. It is well known that differential equations, integral equations and integral-differential equations have gained considerable importance and attention due to their applications in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physical, mechanics, chemistry, aerodynamics, and the electrodynamics of complex mediums, etc. Gronwall-Bellman inequality is an important tool in the study of existence, uniqueness, boundedness, and other qualitative properties of solutions of differential equations and integral equation. In this paper, we discuss we establish a class of retarded iterated integral inequalities, which includes a nonconstant term outside the integrals. By integral inequality technique, the upper bound of the embedded unknown function is estimated explicitly. The derived result can be applied in the study of solutions of fractional integral equations.

Introduction

It is well known that differential equations and integral equations have gained considerable importance and attention due to their applications in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physical, mechanics, chemistry, aerodynamics, and the electrodynamics of complex mediums, etc. In the study of the existence, uniqueness, boundedness, stability, continuous dependence on initial data and parameters of solutions, oscillation and other qualitative properties of solutions of differential equations and integral equations, one often deals with certain iterated integral inequalities. In 1919, Gronwall [1,2] introduced the famous Gronwall inequality, which can be stated as follows: If u and f are non-negative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds,\ t \in [a,b], \tag{1}$$

for some constant $c \ge 0$, then

$$u(t) \le c\exp\left(\int_a^t f(s)ds\right), t \in [a,b].$$

In 2011, Abdeldaim et al. [3] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t) \le u_0 + \int_0^t f(s)u(s)[u(s) + \int_0^s h(\tau)[u(\tau) + \int_0^\tau g(\xi)d\xi]d\tau]ds\,. \tag{2}$$

In 2014, El-Owaidy, Abdeldaim, and El-Deeb [4] investigated a new retarded nonlinear integral inequality

$$u(t) \le f(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^p(s)ds\,. \tag{3}$$

In 2014, Zheng [5] discussed the inequality of the following form

$$u(t) \le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)u(s)d\,. \tag{4}$$

In this paper, on the basis of [3, 4,6], we discuss a new iterated nonlinear integral inequality

$$u(t) \le f(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)u(s)[u(s)$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h(\tau) u(\tau) d\tau] ds. \tag{5}$$

Result

Throughout this paper, let $R_+ = [0, +\infty)$.

The modified Riemann-Liouville derivative and fractional integral, presented by Jumarie in [7,8], is defined by the following expression.

Definition 1. The modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, \ 0 < \alpha < 1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, \ n < \alpha < n+1, n \geq 1. \end{cases} \tag{6}$$

Definition 2.The Riemann-Liouville fractional integral of order α on the interval is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(1+\alpha)} \int_0^t f(s)(ds)^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{7}$$

In 2014, Zheng [5] proved the following property

Lemma 1. Suppose that $0 < \alpha < 1$, f is a continuous function, then

$$D_t^\alpha \left(I^\alpha f(t)\right) = f(t). \tag{8}$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows (see [9, 10]).

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{9}$$

$$D_t^\alpha (f(t) g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t). \tag{10}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)] D_t^\alpha g(t) = D_t^\alpha f[g(t)] (g'(t))^\alpha, \tag{11}$$

$$I^\alpha \left(D_t^\alpha f(t)\right) = f(t) - f(0), \tag{12}$$

$$D_t^\alpha C = 0, \quad \text{where } C \text{ is a constant.} \tag{13}$$

Theorem 1. Suppose that $g(t),\ h(t) \in C(R_+, R_+),\ \ f \in (R_+, R_+)$ is a nondecreasing function with $f(t) > 0$ for $t > 0$. Suppose that

$$1 - \left\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \exp\left[-\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right] ds\right] g(\tau) f(\tau) d\tau\right\}^{-1} > 0. \tag{14}$$

If $u(t)$ satisfies (5), then

$$u(t) \leq f(t) \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right] ds\right] \{1$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \exp\left[-\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right] ds\right] g(\tau) f(\tau) d\tau\Big\}^{-1}. \tag{15}$$

Proof. Since f (t) is a positive and nondecreasing function. From (5) we obtain

$$\frac{u(t)}{f(t)} \leq 1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \frac{u(s)}{f(s)} ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) f(s) \frac{u(s)}{f(s)} \left[\frac{u(s)}{f(s)}\right.$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} h(\tau) \frac{u(\tau)}{f(\tau)} d\tau\right] ds. \tag{16}$$

Let $v(t) = u(t)/f(t)$. We have

$$v(t) \le 1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)v(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)f(s)v(s)\Big[v(s)$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h(\tau)v(\tau)d\tau\Big]ds. \tag{17}$$

Define a function $z(t)$ by the right hand side of the above inequality

$$z(t) = 1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)v(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)f(s)v(s)\Big[v(s)$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h(\tau)v(\tau)d\tau\Big]ds, \tag{18}$$

which is a positive and nondecreasing function on R_+. From (17) and (18) we have

$$v(t) \le z(t), t \in R_+, \tag{19}$$
$$z(0) = 1. \tag{20}$$

By use of Lemma 1, using (19) we have

$$D_t^\alpha z(t) = g(t)v(t) + g(t)f(t)v(t)\left\{v(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau)v(\tau)d\tau\right\}$$
$$\le g(t)z(t) + g(t)f(t)z(t)\left\{z(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau)z(\tau)d\tau\right\}$$
$$= g(t)z(t)[1 + f(t)w(t)], \tag{21}$$

where

$$w(t) = z(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau)z(\tau)d\tau, \tag{22}$$

which is a positive and nondecreasing function on R_+. From (20) and (22) we have

$$z(t) \le w(t), t \in R_+, \tag{23}$$
$$w(0) = z(0) = 1. \tag{24}$$

Using (21) and (23), we have

$$D_t^\alpha w(t) = D_t^\alpha z(t) + h(t)z(t) \le g(t)z(t)[1 + f(t)w(t)] + h(t)z(t)$$
$$\le g(t)w(t)[1 + f(t)w(t)] + h(t)w(t) \le g(t)f(t)w^2(t) + [g(t) + h(t)]w(t), \tag{25}$$

Since $w(t) > 0$, from (25) we have

$$D_t^\alpha\left(-(w(t))^{-1}\right) \le [g(t) + h(t)](w(t))^{-1} + g(t)f(t). \tag{26}$$

Let $x(t) = -(w(t))^{-1}$, then $x(0) = -(w(0))^{-1} = -1$, from (26) we get

$$D_t^\alpha x(t) - [g(t) + h(t)]x(t) \le g(t)f(t). \tag{27}$$

On other hand, using the properties (9), (10), (11), we have

$$D_t^\alpha\left\{x(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right]ds\right]\right\}$$
$$= \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right]ds\right]D_t^\alpha x(t)$$
$$- x(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right]ds\right][g(t) + h(t)]D_t^\alpha\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)$$
$$= \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) + h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right]ds\right]\left[D_t^\alpha x(t) - [g(t) + h(t)]x(t)\right]$$

$$\leq \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)+h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)ds\right]g(t)f(t)\right]. \tag{28}$$

Substituting t with τ in (28), making a fractional integral of order α for (28) with respect to τ from 0 to t and using the properties (12), we obtain that

$$x(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)+h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right]ds\right]-x(0)$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\left[g\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)+h\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right)\right]ds\right]g(\tau)f(\tau)d\tau \tag{29}$$

From (19), (23) and (29), we obtain the required estimation (15). The proof is complete.

Summary

In this paper, we discuss we establish a class of retarded iterated integral inequalities. Using several properties for the modified Riemann-Liouville derivative and fractional integral, the upper bound of the embedded unknown function is estimated explicitly.

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