Stability Analysis of a Type of Takagi-Sugeno PI Fuzzy Control Systems Using Circle Criterion

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Abstract

A type of Takagi-Sugeno (T-S) Proportional-Integral (PI) fuzzy controllers is studied. The T-S PI fuzzy controller is formed by a T-S Proportional-Derivative (PD) fuzzy controller connected with an integrator. In this particular structure, the T-S PD fuzzy controller is a weighted sum of some linear PD sub-controllers. The mathematical properties of our T-S PI fuzzy controller are also investigated. Based on these properties, the global asymptotic stability of the fuzzy control systems, in which the T-S PI fuzzy controllers are employed, are analyzed by using the well-known circle criterion. A sufficient condition with an elegant graphical interpretation in the frequency domain is further derived to guarantee the global asymptotic stability of the above fuzzy control systems. Finally, two numerical examples are provided to demonstrate how to deploy this method in analyzing the T-S PI fuzzy control systems in the frequency domain with the aid of some simple graphs.

Keywords: fuzzy control systems, Takagi-Sugeno (T-S) PI fuzzy controllers, stability analysis, circle criterion, frequency response methods.

1. Introduction

Despite that numerous sophisticated control theories and techniques have been developed in the last decades, the Proportional-Integral-Derivative (PID) controllers are still being applied in more than 90% of the industrial control loops, because they can offer satisfactory control performances at the acceptable costs [1, 2]. The most famous technique for tuning the parameters of the PID controllers is the Eiegler-Nichols method [3], which is typically used as a comparison reference for new tuning schemes. Panagopoulos *et al.* [4] explore the relationship

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between the PID design and H_∞ loop shaping. The specifications in terms of the maximum sensitivity and maximum complementary sensitivity are related to the weighted H_∞ norm. The Bode's integral is also applied to adjust the PID parameters [5]. Based on numerical optimization, a kind of simple but robust PI and PID controllers design approach is presented [6], where the effectiveness of this method is demonstrated by simulations. For more PID tuning strategies, refer to Refs. 1, 2, 7, 8, and references therein.

Fuzzy sets and systems were introduced by Zadeh more than four decades ago. Since then, they have been deployed in a large variety of applications, such as control engineering [9], decision making [10, 11], pattern recognition [12], clustering [13], etc. As a matter of fact, Fuzzy Logic Control (FLC) is one of the most important applications in control theory. As Åström [2] points out that fuzzy control is a necessary alternative for the conventional PID controllers, because the parameterization based on rules and fuzzy membership functions makes it easy to add the knowledge of experts into the control laws. Sugeno [14] classifies the current fuzzy control systems into three major types, i.e., Type I, Type II, and Type III. The Type I, proposed by Mamdani [15], is characterized by a set of fuzzy rules, which are constructed by linguistic terms in both the antecedent and consequent parts of fuzzy rules. The Type II fuzzy systems are derived by using singletons, which are a special class of fuzzy sets, instead of the fuzzy variables in the consequent part of fuzzy rules. Therefore, the Type II is considered to belong to the Type I. If the consequent part of fuzzy rules becomes an analytical function that is generally linear or affine, the Type III fuzzy model can be constructed. The Type III fuzzy model, namely T-S fuzzy model, was proposed by Takagi and Sugeno [16] in 1985. The fuzzy controllers used in industrial applications are usually in the form of fuzzy PD, PI, and PID. However, unlike the conventional PID control systems, the stability analysis of the fuzzy control systems is still an open issue. In Refs. 17-19, the Popov criterion, circle criterion, and describing function method have been respectively utilized to analyze the stability of the Mamdani PI and PD fuzzy controllers with linear plants. For the stability analysis of the Mamdani fuzzy controllers with nonlinear plants, the small gain theorem and Lyapunov stability theorem are considered [20-22].

As for the T-S fuzzy systems, the stability analysis related topics have been investigated [9, 23, 24]. Unfortunately, there are yet not enough research results for the T-S PID control systems. Two sufficient stability conditions with insightful graphic interpretations are derived to guarantee the global stability of the T-S proportional fuzzy control systems by using the circle criterion and Popov criterion, respectively [25, 26]. With the aid of the small gain theorem, Ding et al. analyze the stability of the T-S PI and PD fuzzy controllers with nonlinear plants 27. The T-S PI fuzzy controller is applied to control the integral plants with parameter uncertainty [28] on the basis of a polynomial theorem, which is proposed and studied by Kharitonov [29]. In this paper, we present a sufficient stability condition for a type of the T-S PI fuzzy control systems with a concise graphical interpretation by using the circle criterion. Nevertheless, the condition is derived under some assumptions on the parameters of the consequent parts of the fuzzy logic rules. The rest of our paper is organized as follows. In Section 2, the configuration of the T-S PI fuzzy control systems under investigation is first defined, and the properties of these controllers are next investigated. Based on the circle criterion, the global asymptotic stability condition with an insightful graphical interpretation for the T-S PI fuzzy control systems is derived in Section 3. Two numerical simulation examples are further given to verify the effectiveness of the proposed frequency domain-based analysis method in Section 4. Finally, a few conclusions and remarks are drawn in Section 5.

2. The T-S PI Fuzzy Control System

In this section, the structure of the T-S PI fuzzy control system studied in this paper is formally defined. Some restrictions concerning the consequent parts of the fuzzy logic rules are also given for the derivation of our stability theorems.

2.1. *Structure of the T-S PI fuzzy control system*

The structure of the T-S PI fuzzy control system is shown in Fig. 1, where r is the reference input, τ_1, τ_2, and τ_3 are denoted as the constant scaling factors given by experienced operators, the PDFLC is a T-S PD fuzzy controller, e and $\dot{e}$ represent the control error and derivative of the control error, respectively, u is the output of the T-S PI Fuzzy Logic Controller (PIFLC), and $G(s)$ denotes the transfer function of the linear plant.

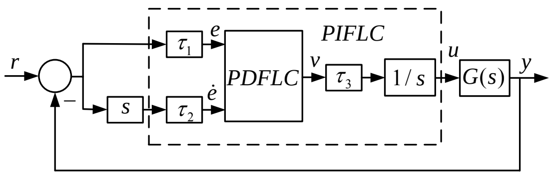

Fig. 1. The T-S PI fuzzy control system.

Obviously, the PIFLC is a PD fuzzy controller in conjunction with an integrator. Each input universe is partitioned into seven fuzzy sets, which are characterized by the commonly used triangle membership functions. The fuzzy sets of e and $\dot{e}$ are denoted as E_i and DE_j, whose corresponding membership functions are respectively represented by $\mu_i(e)$ and $\mu_j(\dot{e})$ (see Fig. 2), where $i, j = -3, -2, \cdots, 3$.

Since there are seven fuzzy sets for each input variable, we need a total of $7 \times 7 = 49$ fuzzy logic rules in order to cover all the combinations of these fuzzy sets. The following T-S fuzzy logic rules are employed in our PDFLC:

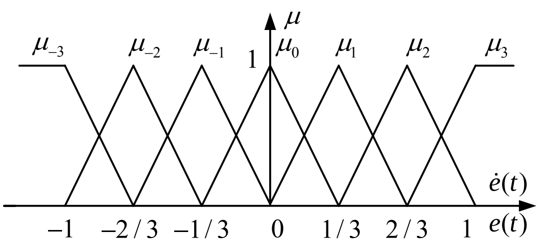

Fig. 2. Membership functions of E_i and DE_i, $i = -3, \cdots, 3$.

Rule $R_{i,j}$: If $e(t)$ is E_i and $\dot{e}(t)$ is DE_j, then

$$v_{i,j} = a_{i,j} e(t) + b_{i,j} \dot{e}(t), \; i, j = -3, \cdots, 3. \tag{1}$$

The common product operator is used in this paper. Thus, the output of the PDFLC is given as follows:

$$v = \frac{\sum_{i=-3}^{3}\sum_{j=-3}^{3} \mu_i(e)\mu_j(\dot{e})(a_{i,j}e + b_{i,j}\dot{e})}{\sum_{i=-3}^{3}\sum_{j=-3}^{3} \mu_i(e)\mu_j(\dot{e})} = \sum_{i=-3}^{3}\sum_{j=-3}^{3} \mu_i(e)\mu_j(\dot{e})(a_{i,j}e + b_{i,j}\dot{e}), \tag{2}$$

and the output of the PIFLC is the integral of $v(t)$, *i.e.*,

$$u(t) = \tau_3 \int_0^t v(s)ds. \tag{3}$$

2.2. *Properties of the control surface of the PDFLC*

In this subsection, some properties of the aforementioned T-S PI fuzzy controller are investigated. We first demonstrate that the binary function achieved by the PDFLC in Fig. 1 is continuous.

Definition 1 [30]: A binary function $f(x, y)$ is said to be continuous at the point (x_0, y_0), if for any $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$, such that

$$\left| f(x, y) - f(x_0, y_0) \right| < \varepsilon, \tag{4}$$

provided that $|x - x_0| < \delta_1$ and $|y - y_0| < \delta_2$. $f(x, y)$ is continuous, if it is continuous at every point in its domain.

Theorem 1: The control surface of the PDFLC in Fig. 1 is continuous.

Proof. Let $v(e, \dot{e})$ denote the binary function achieved by the PDFLC in Fig. 1. We can prove that $v(e, \dot{e})$ is continuous. For any point $(e, \dot{e})$ and its perturbation $(e + \Delta e, \dot{e} + \Delta\dot{e})$, we have

$$v(e, \dot{e}) = \sum_{i=-3}^{3}\sum_{j=-3}^{3} \mu_i(e)\mu_j(\dot{e})(a_{i,j}e + b_{i,j}\dot{e}), \tag{5}$$

$$v(e + \Delta e, \dot{e} + \Delta\dot{e}) = \sum_{i=-3}^{3}\sum_{j=-3}^{3} \mu_i(e + \Delta e)\mu_j(\dot{e} + \Delta\dot{e}) \times [a_{i,j}(e + \Delta e) + b_{i,j}(\dot{e} + \Delta\dot{e})]. \tag{6}$$

From Fig. 2, it can be observed that each membership function $\mu_i(e)$ (or $\mu_i(\dot{e})$), $i = -3, \cdots, 3$, is continuous. Thus, $\mu_i(e)\mu_j(\dot{e})$, $i, j = -3, \cdots, 3$, are continuous, due to the fact that the multiplication of two continuous functions is also continuous. Let

$$A = \max\{|a_{-3}|, \cdots, |a_3|, |b_{-3}|, \cdots, |b_3|\}, \tag{7}$$

we have

$$\left|v(e+\Delta e,\dot e+\Delta\dot e)-v(e,\dot e)\right|$$

$$=\left|\begin{array}{l}\sum_{i=-3}^{3}\sum_{j=-3}^{e}\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)[a_{i,j}(e+\Delta e)+b_{i,j}(\dot e+\Delta\dot e)]\\ -\sum_{i=-3}^{3}\sum_{j=-3}^{3}\mu_i(e)\mu_j(\dot e)(a_{i,j}e+b_{i,j}\dot e)\end{array}\right|$$

$$=\left|\begin{array}{l}\sum_{i=-3}^{3}\sum_{j=-3}^{e}\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)[a_{i,j}(e+\Delta e)+b_{i,j}(\dot e+\Delta\dot e)]\\ -\sum_{i=-3}^{3}\sum_{j=-3}^{3}\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)(a_{i,j}e+b_{i,j}\dot e)\\ +\sum_{i=-3}^{3}\sum_{j=-3}^{3}\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)(a_{i,j}e+b_{i,j}\dot e)\\ -\sum_{i=-3}^{3}\sum_{j=-3}^{3}\mu_i(e)\mu_j(\dot e)(a_{i,j}e+b_{i,j}\dot e)\end{array}\right|$$

$$=\left|\begin{array}{l}\sum_{i=-3}^{3}\sum_{j=-3}^{3}\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)(a_{i,j}\Delta e+b_{i,j}\Delta\dot e)+\\ \sum_{i=-3}^{3}\sum_{j=-3}^{3}[\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)-\mu_i(e)\mu_j(\dot e)](a_{i,j}e+b_{i,j}\dot e)\end{array}\right|$$

$$\le A(|e|+|\dot e|)\sum_{i=-3}^{3}\sum_{j=-3}^{3}\left|\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)-\mu_i(e)\mu_j(\dot e)\right|+ 49A(|\Delta e|+|\Delta\dot e|). \tag{8}$$

Since $\mu_i(e)\mu_j(\dot e)$ is continuous, for $\frac{\varepsilon}{98A(|e|+|\dot e|)}>0$, there exists $\delta_{1,ij},\delta_{2,ij}>0$, such that

$$\left|\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)-\mu_i(e)\mu_j(\dot e)\right|<\frac{\varepsilon}{98A(|e|+|\dot e|)}, \tag{9}$$

when $|\Delta e|<\delta_{1,ij}$ and $|\Delta\dot e|<\delta_{2,ij}$, where $i,j=-3,\cdots,3$. Let

$$\delta_1=\min\{\frac{\varepsilon}{196A},\delta_{1,ij}\mid i,j=-3,\cdots,3\}, \tag{10}$$

and

$$\delta_2=\min\{\frac{\varepsilon}{196A},\delta_{2,ij}\mid i,j=-3,\cdots,3\}, \tag{11}$$

and substitute (9)-(11) into (8). There is

$$\left|v(e+\Delta e,\dot e+\Delta\dot e)-v(e,\dot e)\right|$$

$$\le A(|e|+|\dot e|)\sum_{i=-3}^{3}\sum_{j=-3}^{3}\left|\mu_i(e+\Delta e)\mu_j(\dot e+\Delta\dot e)-\mu_i(e)\mu_j(\dot e)\right|+ 49A(|\Delta e|+|\Delta\dot e|)$$

$$\le 49A(2\times\frac{\varepsilon}{196A})+A(|e|+|\dot e|)\times 49\times\frac{\varepsilon}{98A(|e|+|\dot e|)} \tag{12}$$

$$=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

The above analysis proves that for arbitrary point $(e,\dot e)$ and $\varepsilon>0$, there exists $\delta_1,\delta_2>0$, such that $|v(e+\Delta e,\dot e+\Delta\dot e)-v(e,\dot e)|<\varepsilon$, when $|\Delta e|<\delta_1$ and $|\Delta\dot e|<\delta_2$. Therefore, according to Definition 1, the control surface of the PDFLC is continuous. ■

Remark 1. As we know that the parameters of the consequent parts of the fuzzy rules significantly affect the overall performance of the PI fuzzy controller, and their values should be empirically chosen for different plants.

Theorem 2: If the parameters of the consequent parts of the fuzzy logic rules satisfy the following equalities:

$$a_{i,j}=b_{-j,-i},\ \ i,j=-3,\cdots,3,|i+j|\le 1, \tag{13}$$

the output of the PDFLC is zero, when the sum of the inputs e and $\dot e$ is zero. In other words, $v=0$, when $e+\dot e=0$.

Proof. When $-1<e<1$, four fuzzy logic rules are fired for the combination of e and $\dot e$. Since $e+\dot e=0$, we assume that these four rules are $R_{m,-(m+1)}$, $R_{m,-m}$, $R_{m+1,-(m+1)}$, and $R_{m+1,-m}$, where $m=-3,\cdots,2$. The output of this PDFLC is

$$v=\frac{\sum_{i=m}^{m+1}\sum_{j=-(m+1)}^{-m}\mu_i(e)\mu_j(\dot e)(a_{i,j}e+b_{i,j}\dot e)}{\sum_{i=m}^{m+1}\sum_{j=-(m+1)}^{-m}\mu_i(e)\mu_j(\dot e)} \tag{14}$$

$$=\sum_{i=m}^{m+1}\sum_{j=-(m+1)}^{-m}\mu_i(e)\mu_j(\dot e)(a_{i,j}e+b_{i,j}\dot e).$$

When $e=-\dot e$, we have

$$\mu_m(e)=\mu_{-m}(\dot e), \tag{15}$$

$$\mu_{m+1}(e)=\mu_{-(m+1)}(\dot e). \tag{16}$$

Substituting (13), (15), and (16) into (14), we get

$$v=\mu_m(e)\mu_{-(m+1)}(\dot e)[(a_{m,-(m+1)}e+b_{m+1,-m}\dot e)+ (a_{m+1,-m}e+b_{m,-(m+1)}\dot e)]+ \mu_m^2(e)(a_{m,-m}e+b_{m,-m}\dot e)+$$

$$
\begin{aligned}
&\mu_{m+1}^2(e)(a_{m+1,-(m+1)}e+b_{m+1,-(m+1)}\dot{e})\\
&=\mu_m(e)\mu_{-(m+1)}(\dot{e})[(a_{m,-(m+1)}e-b_{m+1,-m}\dot{e})+\\
&(a_{m+1,-m}e-b_{m,-(m+1)}\dot{e})]+\\
&\mu_m^2(e)(a_{m,-m}e-b_{m,-m}\dot{e})+\\
&\mu_{m+1}^2(e)(a_{m+1,-(m+1)}e-b_{m+1,-(m+1)}\dot{e})\\
&=0.
\end{aligned}
\quad (17)
$$

When $e \le -1$ or $e \ge 1$, there is only one rule fired for the combination of e and $\dot{e}$, which is $R_{-3,3}$ or $R_{3,-3}$. Thus, the output of the PDFLC is

$$
\begin{aligned}
v&=\frac{\mu_{-3}(e)\mu_3(\dot{e})(a_{-3,3}e+b_{-3,3}\dot{e})}{\mu_{-3}(e)\mu_3(\dot{e})}\\
&=(a_{-3,3}-b_{-3,3})e\\
&=0,
\end{aligned}
\quad (18)
$$

or

$$
\begin{aligned}
v&=\frac{\mu_3(e)\mu_{-3}(\dot{e})(a_{3,-3}e+b_{3,-3}\dot{e})}{\mu_3(e)\mu_{-3}(\dot{e})}\\
&=(a_{3,-3}-b_{3,-3})e\\
&=0.
\end{aligned}
\quad (19)
$$

Obviously, $v=0$, because $e+\dot{e}=0$. ■

Remark 2. We must emphasize that the above assumption (13) is not a stringent requirement but a good choice for guaranteeing satisfactory system performances. For example, when e is negative big but $\dot{e}$ is positive big means that e approaches zero rapidly, although e is far away from zero. This trend is desirable, and $v=0$ is appropriate for the control actions. When e is positive small and $\dot{e}$ is negative small implies that e approaches zero slowly. When e is small and $v=0$ is still suitable means that the control actions of the PIFLC are maintained and large overshoots and undershoots can be effectively avoided. The same analysis also applies to the other cases, when $e+\dot{e}=0$.

Theorem 3: If the conditions of Theorem 2 are satisfied, and $v(e,\dot{e})$ is denoted as the nonlinear function achieved by the PDFLC, there exist two real numbers k_1 and k_2 satisfying the following inequality:

$$k_1(e+\dot{e})^2 \le v(e,\dot{e})(e+\dot{e}) \le k_2(e+\dot{e})^2. \quad (20)$$

Proof. In region A (as shown in Fig. 3), where $e \le -1$ and $\dot{e} \ge 1$, the output of our PDFLC is

$$v(e,\dot{e})=a_{-3,3}e+b_{-3,3}\dot{e}=a_{-3,3}(e+\dot{e}). \quad (21)$$

There is

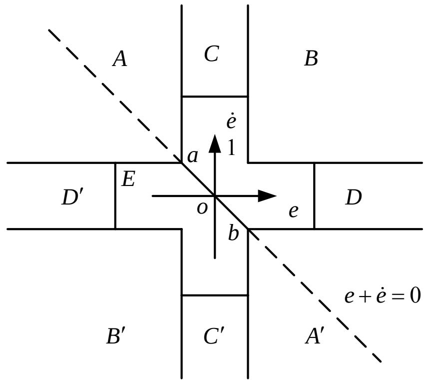

Fig. 3. Partition of input space.

$$|v(e,\dot{e})|=|a_{-3,3}||e+\dot{e}|. \quad (22)$$

Similarly, we have $|v(e,\dot{e})|=|a_{3,-3}||e+\dot{e}|$ in region A', where $e \ge 1$ and $\dot{e} \le -1$.

In region B, where $e \ge 1$ and $\dot{e} \ge 1$, the output of the PDFLC is $v(e,\dot{e})=a_{3,3}e+b_{3,3}\dot{e}$. Therefore, there is

$$|v(e,\dot{e})| \le \max\{|a_{3,3}|,|b_{3,3}|\}|e+\dot{e}|. \quad (23)$$

Similarly, we have $|v(e,\dot{e})| \le \max\{|a_{-3,-3}|,|b_{-3,-3}|\}|e+\dot{e}|$ in region B', where $e \le -1$ and $\dot{e} \le -1$.

In region D, where $|\dot{e}|<1$ and $e>3$, there is

$$
\begin{aligned}
v(e,\dot{e})&=\sum_{k=i}^{i+1}\mu_3(e)\mu_k(\dot{e})(a_{3,k}e+b_{3,k}\dot{e})\\
&=\sum_{k=i}^{i+1}\mu_k(\dot{e})(a_{3,k}e+b_{3,k}\dot{e}), i=-3,\cdots,2.
\end{aligned}
\quad (24)
$$

Hence, we have

$$
\begin{aligned}
|v(e,\dot{e})|&=\left|\sum_{k=i}^{i+1}\mu_k(\dot{e})(a_{3,k}e+b_{3,k}\dot{e})\right|\\
&\le \sum_{k=i}^{i+1}|a_{3,k}e+b_{3,k}\dot{e}|=\sum_{k=i}^{i+1}\left|\frac{a_{3,k}e+b_{3,k}\dot{e}}{e+\dot{e}}\right||e+\dot{e}|\\
&\le \sum_{k=i}^{i+1}(|a_{3,k}|+|b_{3,k}|)|e+\dot{e}|\\
&\le \max\{\sum_{k=i}^{i+1}(|a_{3,k}|+|b_{3,k}|): i=-3,\cdots,2\}|e+\dot{e}|.
\end{aligned}
\quad (25)
$$

In region D', where $|\dot{e}|<1$ and $e<-3$, we get

$$|v(e,\dot{e})| \le \max\{\sum_{k=i}^{i+1}(|a_{-3,k}|+|b_{-3,k}|): i=-3,\cdots,2\}|e+\dot{e}|. \quad (26)$$

We also have

$$|v(e,\dot{e})| \le \max\{\sum_{k=i}^{i+1}(|a_{k,3}|+|b_{k,3}|): i=-3,\cdots,2\}|e+\dot{e}|, \quad (27)$$

in region C and

$$|v(e,\dot{e})| \le \max\{\sum_{k=i}^{i+1}(|a_{k,-3}|+|b_{k,-3}|): i=-3,\cdots,2\}|e+\dot{e}|, \quad (28)$$

in region C'.

Actually, the analysis of region E, where $E=([-1,1]\times[-3,3])\bigcup([-3,3]\times[-1,1])$, is more complicated than the other cases. Line segment ab is the intersection of $e+\dot{e}=0$ and E (see Fig. 3). Our goal here is to prove that $|v(e,\dot{e})/(e+\dot{e})|$ is bounded in E, when $e+\dot{e}\ne 0$. We first divide region E into 60 small regions, such as E_1', E_1'', and E_1, as shown in Fig. 4, and use the $x-y$ coordinate to replace $e-\dot{e}$ coordinate.

Thus, we obtain

$$x=\sqrt{2}(e+\dot{e}), \quad (29)$$

$$y=\sqrt{2}(\dot{e}-e). \quad (30)$$

It is straightforward to observe that $v(e,\dot{e})$ is analytical in each small region in Fig. 4.

For example, the mathematical expression of $v(e,\dot{e})$ in E_1' is as follows:

$$\begin{aligned} v(e,\dot{e}) &= \sum_{i=2}^{3}\mu_i(\dot{e})(a_{-3,i}e+b_{-3,i}\dot{e}) \\ &= 3(b_{-3,3}-b_{-3,2})\dot{e}^2+3(a_{-3,3}-a_{-3,2})e\dot{e}+ \\ &\quad (3a_{-3,2}-2a_{-3,3})e+(3b_{-3,2}-2b_{3,3})\dot{e}. \end{aligned} \quad (31)$$

Therefore, in the $x-y$ coordinate, each point on the line segment ab is analytical except for $\{(x,y)\,|\,x=0, y=-2\sqrt{2},-\frac{4\sqrt{2}}{3},\cdots,2\sqrt{2}\}$. This means $\left|\frac{\partial v(x,y)}{\partial x}\Big|_{x=0}\right|$, $y\ne -2\sqrt{2},-\frac{4\sqrt{2}}{3},\cdots,2\sqrt{2}$, exists, and is bounded. There is

$$\begin{aligned} \lim_{x\to 0}\left|\frac{v(x,y)}{x}\right| &= \lim_{x\to 0}\left|\frac{v(x,y)-v(0,y)}{x-0}\right| \\ &= \left|\frac{\partial v(x,y)}{\partial x}\Big|_{x=0}\right| \\ &\le M_1', \end{aligned} \quad (32)$$

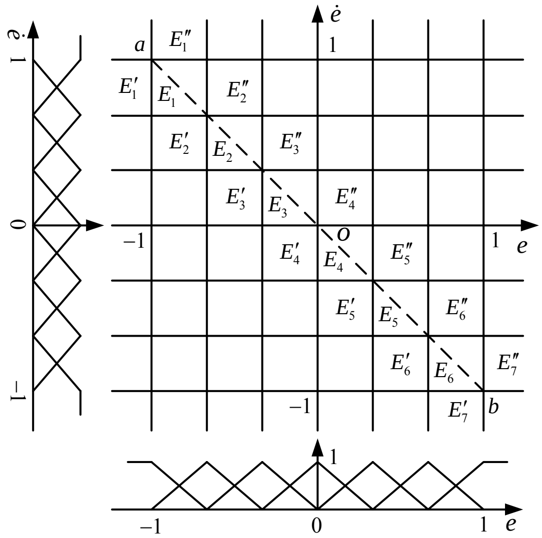

Fig. 4. Partition of region E.

where, $|y|\le 2\sqrt{2}$, and $y\ne -2\sqrt{2},-\frac{4\sqrt{2}}{3},\cdots,2\sqrt{2}$. Yet point (x,y), where $x=0$ and $y=-2\sqrt{2},-\frac{4\sqrt{2}}{3},\cdots,2\sqrt{2}$, is not analytical, because it is on the edge of four different small regions. However,

$$\lim_{x\to 0^+}\frac{v(x,y)-v(0,y)}{x-0}=\lim_{x\to 0^+}\frac{v(x,y)}{x}, \quad (33)$$

and

$$\lim_{x\to 0^-}\frac{v(x,y)-v(0,y)}{x-0}=\lim_{x\to 0^-}\frac{v(x,y)}{x}, \quad (34)$$

indeed exist, due to the fact that $(0^+,y)$ and $(0^-,y)$ are analytical in different regions. There also exists $M_1''>0$, such that

$$\lim_{x\to 0}\left|\frac{v(x,y)}{x}\right|\le M_1'', \quad (35)$$

where $y=-2\sqrt{2},-\frac{4\sqrt{2}}{3},\cdots,2\sqrt{2}$.

With (32) and (35), we obtain

$$\begin{aligned} \lim_{e+\dot{e}\to 0}\left|\frac{v(e,\dot{e})}{e+\dot{e}}\right| &= \lim_{x\to 0}\left|\frac{v(x,y)}{x}\right| \\ &\le \max\{M_1',M_1''\},(e,\dot{e})\in E. \end{aligned} \quad (36)$$

Obviously, there is $\delta>0$, such that

$$|v(e,\dot{e})|\le \max\{M_1',M_1''\}|e+\dot{e}|, \quad |e+\dot{e}|<\delta. \quad (37)$$

For the point belonging to $E \backslash |e+\dot{e}| < \delta$, $v(e,\dot{e})$ is finite, because $v(e,\dot{e})$ is continuous on R^2 (see Theorem 1). Thus, $|v(e,\dot{e})/(e+\dot{e})|$ is bounded. Assume that $M_1''' > 0$ is the boundary. In region E , we have

$$|v(e,\dot{e})| \le M_1 |e+\dot{e}|, \quad M_1 = \max\{M_1', M_1'', M_1'''\}. \tag{38}$$

From the above different cases, we can find an $M > 0$, such that

$$|v(e,\dot{e})| \le M |e+\dot{e}|. \tag{39}$$

Therefore, (20) follows from (39). ■

Remark 3. We would like to point out the following two interesting issues. Firstly, this theorem shows the control surface of our PDFLC can be caught by two planes with finite slopes. Secondly, these slopes are determined by the parameters of the consequent parts of the fuzzy logic rules. Actually, the minimum value of k_2 and maximum value of k_1 satisfying (20) can be theoretically derived.

To summarize, we investigate some crucial properties of a type of fuzzy controllers in this section. For more properties of the general T-S fuzzy controllers, refer to Refs. 9 and 31. The circle criterion will be employed to analyze the stability of the closed-loop fuzzy control systems in the next section.

3. Stability Analysis

The stability analysis of control systems is indeed important. Unlike the other conventional methods for controller design, e.g., sliding model control and adaptive control, in which the stability of the closed-loop systems is taken into consideration in the design process, the design of the fuzzy logic controllers heavily depends on the experienced experts. The advantage is that the operators' domain knowledge can be embedded into the controllers. Unfortunately, this intuitive approach increases the difficulties of stability analysis of the fuzzy control systems. A novel circle criterion-based method is proposed to guarantee the local asymptotic stability of a specific type of the Mamdani PI fuzzy controllers with linear plants [18]. However, it cannot be generalized to a circle criterion-based condition for the global asymptotic stability, due to the saturation of the Mamdani FLCs.

It is well known that many nonlinear systems can be represented as a feedback connection (i.e., Lur'e form) of a linear dynamical system and a nonlinear element. Therefore, the frequency domain-based methods can be employed to analyze the stability of this class of nonlinear systems, such as the circle criterion (refer to **Appendix A** for a brief introduction) and Popov criterion. In the present paper, we only focus on transforming the T-S PI fuzzy control systems into the standard Lur'e form. Since the Lyapunov stability of the closed-loop systems is studied here, the external input r is assumed to be zero. Observe that line $e+\dot{e}=0$ is on the control surface of the PDFLC by Theorem 3, which implies that the control surface is between two planes crossing the line $e+\dot{e}=0$. If $f(e,\dot{e})$ is denoted as a nonlinear coefficient, on the basis of Theorem 3, the output of the PDFLC can be characterized by

$$v(e,\dot{e}) = f(e,\dot{e})(e+\dot{e}), \quad e+\dot{e} \ne 0 . \tag{40}$$

Thus, $f(e,\dot{e})$ belongs to sector $[k_1, k_2]$, and the system in Fig. 1 is recast into its version in Fig. 5. Furthermore, the one in Fig. 5 is transformed into the standard Lur'e form, as illustrated in Fig. 6.

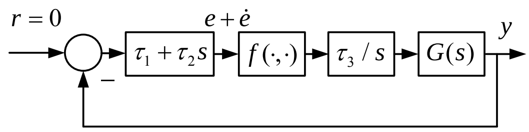

Fig. 5. Equivalent structure of the system in Fig. 1.

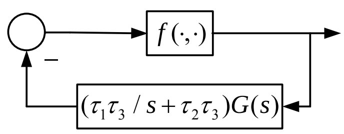

Fig. 6. Standard Lur'e form of the system in Fig. 1.

To conclude, the circle criterion can be used to analyze the stability of the above T-S PI fuzzy control systems, and we can further obtain the following Theorem 4.

Theorem 4: The T-S PI fuzzy control system in Fig. 1 is globally asymptotically stable, if the control surface of the PDFLC is caught by two planes $v = k_1(e+\dot{e})$ and $v = k_2(e+\dot{e})$, where $0 < k_1 < k_2$, and the Nyquist plot of $(\frac{\tau_1\tau_2}{j\omega} + \tau_2\tau_3)G(j\omega)$ does not enter disk $D(k_1,k_2)$ but encircles it p times in the counterclockwise direction. p

denotes the number of the poles of $(\frac{\tau_1\tau_2}{j\omega}+\tau_2\tau_3)G(j\omega)$ with positive real parts, $G(s)$ is the transfer function of the linear plant, and $D(k_1,k_2)$ is defined to be the closed disk in the complex plane, whose diameter is the line segment connecting points $(-1/k_1,0)$ and $(-1/k_2,0)$.

Proof. By transforming the system in Fig. 1 into the standard Lur'e form as well as using the circle criterion, we can prove this theorem in a straightforward way. ■

It should be mentioned that the maximum value of k_1 and minimum value of k_2 satisfying (20) can be theoretically obtained by analyzing each small region in Fig. 3. Nevertheless, if it is difficult to find the boundaries of $\frac{v(e,\dot{e})}{e+\dot{e}}$, we can determine k_1 and k_2 from the lateral view of the control surface of the PDFLC. For example, Figure 7 illustrates the control surface of a given PDFLC. A rotated version of Fig. 7 is shown in Fig. 8, from which it can be observed that $k_1=0$ and $k_2=6$ are indeed the boundaries of the control surface of this PDFLC.

Comparing the control surface of the T-S fuzzy controller in Fig. 8 with that of the Mamdani fuzzy controller in Fig. 9 of Ref. 18, we can discover the saturation phenomenon of the Mamdani fuzzy controller. In Ref. 18, the control surface of the Mamdani PD fuzzy controller is locally caught by two planes $v=k_1(e+\dot{e})$ and $v=k_2(e+\dot{e})$, where $0<k_1<k_2$. By using the first condition of the circle criterion, a sufficient condition is derived for the local asymptotic stability. Actually, the control surface is between two planes, whose slopes are 0 and k_2, due to the saturation. Unfortunately, the second condition of the circle criterion cannot be employed to derive a global asymptotic stability condition, because the linear part of the standard Lur'e form of the closed-loop fuzzy control system is not Hurwitz. However, since the T-S PD fuzzy controller is free from the saturation problem, the circle criterion can be directly utilized in this case.

4. Simulations

In this section, the proposed analysis method is examined to analyze the stability of the T-S PI fuzzy control systems. The T-S PI fuzzy controllers are used to control the

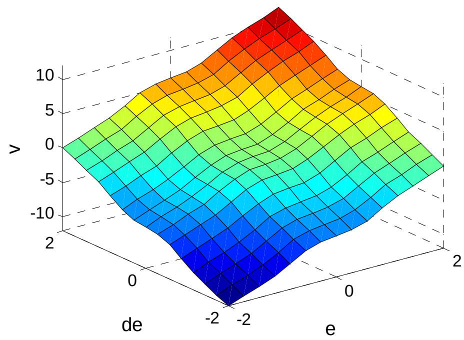

Fig. 7. Control surface of a PDFLC.

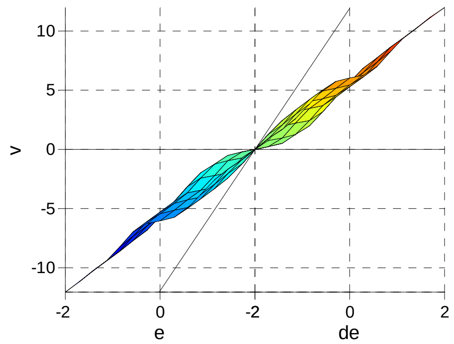

Fig. 8. Control surface of the PDFLC in Fig. 7 observed from azimuth 45° and elevation 0°.

marginally stable and unstable plants in the following two examples, respectively. It should be noted that only the stability analysis is investigated in the present paper, and tuning the controller parameters to achieve perfect performances is not the main goal. Therefore, the T-S fuzzy controller design procedure as well as a detailed performance comparison between the PI fuzzy controller and traditional PID controller is omitted here. As a matter of fact, the controllers in the two examples below are heuristically designed based on the conventional PID tuning methods.

Example 1 [32]: In this example, a marginally stable plant $G(s)$ is described by

$$G(s)=\frac{1}{s(s+1)}, \tag{41}$$

for which even the Eiegler-Nichols-tuned PID controllers fail to yield an acceptable control performance. Therefore, a PIFLC is constructed for $G(s)$, as shown in Fig. 1, where $\tau_1=1$, $\tau_2=4$, and $\tau_3=1$. The control surface of the PDFLC in Fig. 9 is designed to satisfy (20), where

$k_1 = 0.5$ and $k_2 = 2.4$. The two corresponding planes are $v = 0.5(e + \dot{e})$ and $v = 2.4(e + \dot{e})$. Both the Nyquist plot of (41) and disk $D(0.5, 2.4)$ are given in Fig. 10. It follows from Theorem 4 that our fuzzy control system is globally asymptotically stable, whose step response is illustrated in Fig. 11. Obviously, the T-S PI fuzzy controller performs significantly better than the traditional PI controller

Example 2 [19]: In this example, the transfer function of an unstable plant is described by

$$G(s) = \frac{0.1s + 1}{s^2 - s + 1}, \tag{42}$$

which has two poles with positive real parts. A T-S PI fuzzy controller is constructed for the unstable plant, where τ_1, τ_2, and τ_3 are chosen to be $\tau_1 = 1$, $\tau_2 = 0.27$, and $\tau_3 = 352$. The parameters of the consequent parts of the fuzzy logic rules are fine-tuned so that the control surface of our PDFLC is between planes $v = 0.7(e + \dot{e})$ and $v = 5(e + \dot{e})$, as illustrated in Fig. 12. The Nyquist plot of (42) does not enter disk $D(0.7, 5)$, and it encircles the disk $D(0.7, 5)$ twice in the counterclockwise direction. Thus, based on Theorem 4, we can find out that this fuzzy control system is globally asymptotically stable. The Nyquist plot of (42) and disk $D(0.7, 5)$ are shown in Fig. 13. Figure 14 further demonstrates the small region around disk $D(0.7, 5)$ in more details. From Figs. 13 and 14, it is clearly visible that the Nyquist plot indeed encircles the disk $D(0.7, 5)$ twice in the counterclockwise direction, and the global asymptotic stability can be ensured. The step response of the above fuzzy control system is illustrated in Fig. 15, which depicts that the T-S PI fuzzy controller outperforms the regular PI controller.

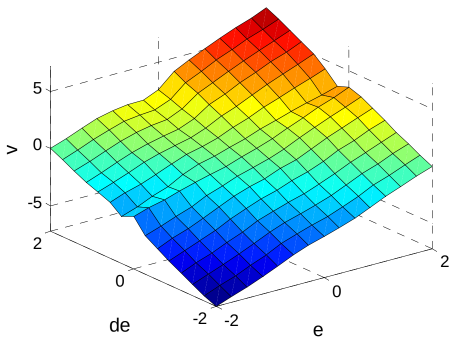

Fig. 9. Control surface of the PDFLC in Example 1.

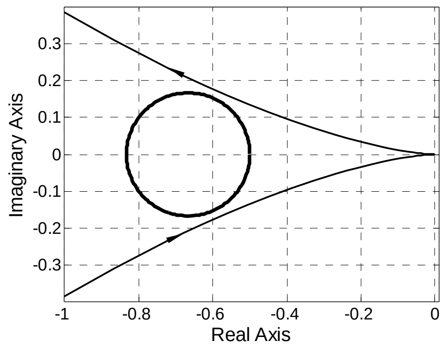

Fig. 10. Nyquist plot of $\frac{4s + 1}{s^2(s + 1)}$ and disk $D(0.5, 2.4)$.

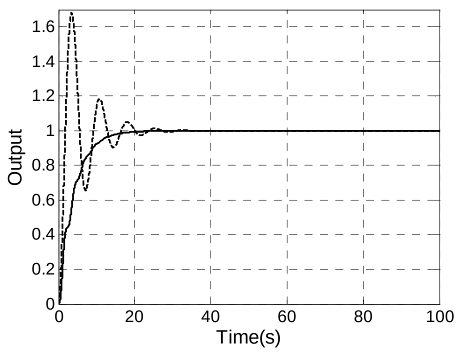

Fig. 11. Step responses of a marginally stable plant with the T-S PI fuzzy controller (solid line) and traditional PI controller (dash line).

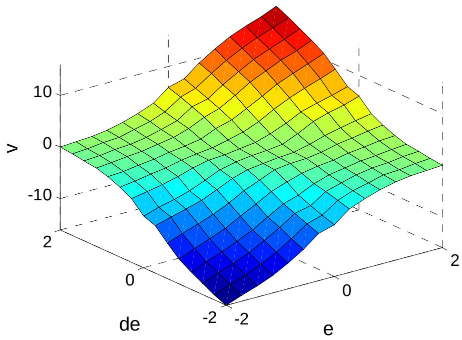

Fig. 12. Control surface of the PDFLC in Example 2.

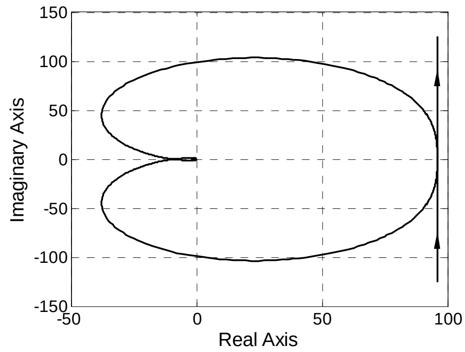

Fig. 13. Nyquist plot of $\frac{(0.1s+1)(95.04s+1)}{s(s^2-s+1)}$ and disk $D(0.7,5)$.

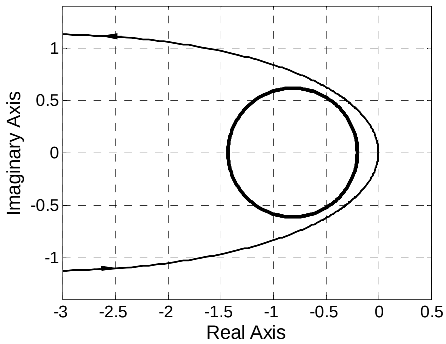

Fig. 14. Disk $D(0.7,5)$ in details.

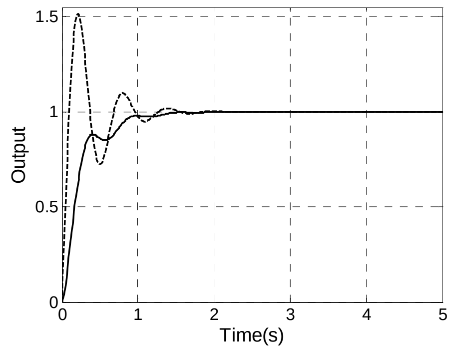

Fig. 15. Step responses of an unstable plant with the T-S PI fuzzy controller (solid line) and traditional PI controller (dash line).

5. Conclusions

In this paper, the circle criterion is employed to graphically analyze the global asymptotic stability of the T-S PI fuzzy control systems in the frequency domain. Two typical numerical examples are provided to demonstrate the efficiency of the proposed approach, which imposes some restrictions on the fuzzy controllers so as to guarantee the global asymptotic stability. However, we only focus on the linear plants here. For the nonlinear plants, the local stability of the closed-loop systems can be also analyzed after linearization. Since a T-S fuzzy model is usually constructed by a family of local linear models, it is indeed interesting and important to analyze the T-S fuzzy models with the T-S fuzzy controllers in the frequency domain. Therefore, in our future work, we are going to further investigate the frequency domain-based stability conditions for the general T-S fuzzy control systems.

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Appendix A.

Let us consider a general nonlinear system given in Fig. 16, which can be described by the following state equations:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\xi(t),\\ \sigma(t) &= \boldsymbol{c}\boldsymbol{x}(t) + d\xi(t),\\ \xi(t) &= -\phi(t,\sigma(t)),\end{aligned} \tag{43}$$

where $\boldsymbol{A} \in R^{n\times n}$, $\boldsymbol{b}, \boldsymbol{c}^T, \boldsymbol{x} \in R^n$, and d, ξ, σ are all scalars. We assume that $(\boldsymbol{A}, \boldsymbol{b})$ pair is controllable, $(\boldsymbol{c}, \boldsymbol{A})$ pair is observable, and $\phi(\cdot,\cdot)$ is a memoryless nonlinear function, which is continuous with respect to its two parameters. We conclude $\phi(\cdot,\cdot)$ belongs to sector $[\alpha, \beta]$, if the following inequality is satisfied:

$$\alpha\sigma^2(t) \le \sigma(t)\phi(t,\sigma(t)) \le \beta\sigma^2(t)\ . \tag{44}$$

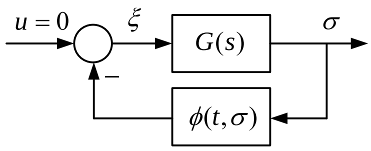

Fig. 16. Nonlinear system analyzed by the circle.

The circle criterion [33]: The system (43) is globally asymptotically stable, if one of the following conditions is satisfied:

1) If $0 < \alpha < \beta$, the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ but encircles it m times in the counterclockwise direction, where m is the number of the poles of $G(s)$ with positive real parts. $D(\alpha, \beta)$ is defined to be a closed disk in the complex plane, whose diameter is the line segment connecting points $-1/\alpha + j0$ and $-1/\beta + j0$, as shown in Fig. 17, where $0 < \alpha < \beta$.

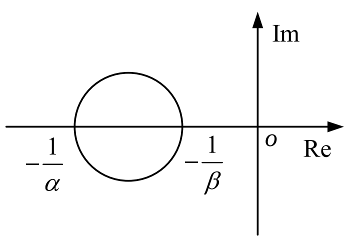

Fig. 17. Illustration of disk $D(\alpha, \beta)$, when $0 < \alpha < \beta$.

2) If $0 = \alpha < \beta$, $G(s)$ is Hurwitz, and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\mathrm{Re}[s] = -1/\beta$.

3) If $\alpha < 0 < \beta$, $G(s)$ is Hurwitz, and the Nyquist plot of $G(j\omega)$ lies in the interior of disk $D(\alpha, \beta)$.

Proof. Refer to Ref. 29 for a detailed proof. ■

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