Estimation of unknown function of a class of nonlinear integral inequality

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Keywords: Gronwall-type inequality; Analysis technique; Explicit bound; Fractional integral equation.

Abstract. In this paper, we establish a class of iterated integral inequality, which includes a noncons -tant term outside the integrals. The upper bound of the embedded unknown function in the inequality is estimated explicitly by adopting novel analytical techniques, such as: change of variable, amplification method, differential and integration. The derived result can be applied in the study of qualitative properties of solutions of fractional integral equations.

Introduction

Gronwall-Bellman inequality [1, 2] can be stated as follows: If u and f are non-negative continuous functions on an interval $[a,b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds, \qquad t \in [a,b]. \tag{1}$$

For some constant $c \ge 0$, then

$$u(t) \le c\exp(\int_a^t f(s)ds), \qquad t \in [a,b].$$

In 2011, Abdeldaim et al. [3] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t) \le u_0 + \int_0^t [g(s)u(s) + q(s)]ds + \int_0^t g(s)u(s)[u(s) + \int_0^s h(\tau)u(\tau)d\tau]ds. \tag{2}$$

In 2014, El-Owaidy, Abdeldaim, and El-Deeb[4] investigated a new retarded nonlinear integral inequality

$$u(t) \le f(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^p(s)ds. \tag{3}$$

In 2014, Zheng [5] discussed the inequality of the following form

$$u(t) \le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^A (A-s)^{\alpha-1} g(s)u(s)ds. \tag{4}$$

During the past few years, some investigators have established a lot of useful and interesting integral inequalities in order to achieve various goals; see [3-10] and the references cited therein.

In this paper, on the basis of [3, 4, 5], we discuss a class of nonlinear weakly singular integral inequality

$$u(t) \le f(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[g(s)u(s) + q(s)f(s)]ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)f(s)u(s)$$

$$\times[u(s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h(\tau)w(u(\tau))d\tau]ds \tag{5}$$

Main result

Throughout this paper, let $R_+ = [0, +\infty)$.

Definition 1(see [7,8]). The modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^{\alpha} f(t)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, 0<\alpha<1\\ (f^{(n)}(t))^{(\alpha-n)}, n<\alpha<n+1, n\geq 1\end{cases} \tag{6}$$

Definition 2(see [7, 8]). The Riemann-Liouville fractional integral of order α on the interval is defined by

$$I_t^{\alpha} f(t)=\frac{1}{\Gamma(1+\alpha)}\int_0^t f(s)(ds)^{\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{7}$$

In 2014, Zheng [5] proved the following property.

Lemma 1. Suppose that $0<\alpha<1$, f is a continuous function, then

$$D_t^{\alpha}(I_t^{\alpha} f(t))=f(t). \tag{8}$$

Some important properties for the modified Riemann-Liouville derivative and frational integral are listed as follows (see [9, 10]):

$$D_t^{\alpha} f[g(t)]=f_g'[g(t)]D_t^{\alpha} g(t)=D_g^{\alpha} f[g(t)](g'(t))^{\alpha}, \tag{9}$$

$$I_t^{\alpha}(D_t^{\alpha} f(t))=f(t)-f(0), \tag{10}$$

$D_t^{\alpha} C=0,$ where C is a constant. (11)

Define three functions by w in (5)

$$W(u)=\int_0^u \frac{\exp(-2\ln(-s))ds}{w\exp(-\ln(-s))}, u\in R_+. \tag{12}$$

Theorem 1.Suppose that, $g,h,q\in C(R_+,R_+)$, $w_3, f\in C(R_+,R_+)$ are nondecreasing functions with $w(u)/v\leq w(u/v)$, $f(u)>0$ for all $u>0, v>0$. If $u(t)$ satisfies (10), then

$$\begin{aligned}u(t)\leq \exp\Bigg\{&-\ln[-W^{-1}(W(-\exp(-\ln(1+\frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\alpha-1}q(s)ds)-\frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\alpha-1}g(s)ds)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds)]\Bigg\} f(t), t\in[0,T_1],\end{aligned} \tag{13}$$

where T_1 is the largest number such that

$$\begin{aligned}&W(-\exp(-\ln(1+\frac{1}{\Gamma(\alpha)}\int_0^{T_1} (T_1-s)^{\alpha-1}q(s)ds)-\frac{1}{\Gamma(\alpha)}\int_0^{T_1} (T_1-s)^{\alpha-1}g(s)ds)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{T_1} (T_1-s)^{\alpha-1}g(s)ds)\in Dom(W^{-1})\end{aligned} \tag{14}$$

Proof. Noting that $f(t)$ is a positive and nondecreasing function, from (5) we obtain

$$\begin{aligned}\frac{u(t)}{f(t)}\leq 1&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)\frac{u(s)}{f(s)}+q(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)\frac{u(s)}{f(s)}[\frac{u(s)}{f(s)}\\ &+\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h(\tau)w\left(\frac{u(\tau)}{f(\tau)}\right)d\tau]ds.\end{aligned} \tag{15}$$

Let $z_1(t)=u(t)/f(t)$. From (14) we see

$$\begin{aligned}z_1(t)\leq 1&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)z_1(s)ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)z_1(s)[z_1(s)+\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h(\tau)w(z_1(\tau))d\tau]ds, t\in R_+.\end{aligned} \tag{16}$$

Define a function $z_2(t)$ by the right hand side of the inequality (21), i.e.

$$z_2(t) = 1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} q(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) z_1(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) z_1(s)\left[z_1(s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h(\tau) w(z_1(\tau)) d\tau\right] ds$$
$$\le 1 + \frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\alpha-1} q(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) z_1(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) z_1(s)\left[z_1(s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h(\tau) w(z_1(\tau)) d\tau\right] ds, t \in [0,T], \quad (17)$$

where $T \in [0, T_1]$ is chosen arbitrarily. We observe that $z_2(t)$ is a positive and nondecreasing function on [0,T]. From (16) and (17) we have

$$z_1(t) \le z_2(t), u(t) \le z_2(t) f(t), t \in [0,T], \quad (18)$$

$$z_2(0) = 1 + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} q(s) ds. \quad (19)$$

Using Lemma 1, the property (8) ,the relation (18),and the definitions of fractional integral and derivative,we get

$$D_t^\alpha z_2(t) = g(t) z_1(t) + g(t) z_1(t)[z_1(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau) w(z_1(\tau)) d\tau]$$
$$\le g(t) z_2(t) + g(t) z_2(t)[z_2(t) + I_t^\alpha (h(t) w(z_2(t)))]$$
$$= g(t) z_2(t)[1 + z_3(t)], t \in [0,T], \quad (20)$$

where

$$z_3(t) = z_2(t) + I_t^\alpha (h(t) w(z_2(t))), \quad (21)$$

which is a positive and nondecreasing function on [0,T]. From (19) and (21) we have

$$z_2(t) \le z_3(t), t \in [0,T], \quad (22)$$

$$z_3(0) = z_2(0) = 1 + \frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\alpha-1} q(s) ds. \quad (23)$$

Using (21) and (22), we have

$$D_t^\alpha z_3(t) = D_t^\alpha z_2(t) + h(t) w(z_2(t)) \le g(t) z_3(t)[1 + z_3(t)] + h(t) w(z_3(t))$$
$$\le g(t) z_3(t) + g(t) z_3^2(t) + h(t) w(z_3(t)), t \in [0,T]. \quad (24)$$

By the formula (9), from (24) we obtain

$$D_t^\alpha [\ln(z_3(t))] = \frac{1}{z_3(t)} D_t^\alpha z_3(t) \le g(t) + g(t) z_3(t) + h(t)\frac{w(z_3(t))}{z_3(t)}, t \in [0,T]. \quad (25)$$

Substituting t with τ in (25), making a fractional integral of order α for (25) with respect to τ from 0 to t and using the properties (10), we obtain

$$\ln(z_3(t)) \le \ln(z_3(0)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) z_3(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\frac{w(z_3(s))}{z_3(s)} ds]$$
$$\le \ln(z_3(0)) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) z_3(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\frac{w(z_3(s))}{z_3(s)} ds, t \in [0,T]. \quad (26)$$

Let z_4 denote the right hand side of the inequality (31), then z_4 is a positive and nondecreasing function on $[0,T]$ with

$$z_3(t) \le \exp(z_4(t)), t \in [0,T], \tag{27}$$

$$z_4(0) = \ln(z_3(0)) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds. \tag{28}$$

Using (27) we have

$$D_t^\alpha(z_4(t)) = g(t)z_3(t) + h(t)\frac{w(z_3(t))}{z_3(t)} \le g(t)\exp(z_4(t)) + h(t)\frac{w(\exp(z_4(t)))}{\exp(z_4(t))}, t \in [0,T]. \tag{29}$$

From (29) we get

$$D_t^\alpha(-\exp(-z_4(t))) = \exp(-z_4(t))D_t^\alpha(z_4(t)) \le g(t) + h(t)\frac{w(\exp(z_4(t)))}{\exp(z_4(t))}, t \in [0,T]. \tag{30}$$

Substituting t with τ in (30), making a fractional integral of order α for (30) with respect to τ from 0 to t and using the properties (10), we obtain that

$$-\exp(-z_4(t)) \le -\exp(-z_4(0)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\frac{w(\exp(z_4(s)))}{\exp(2z_4(s))}ds$$

$$\le -\exp(-z_4(0)) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\frac{w(\exp(z_4(s)))}{\exp(2z_4(s))}ds, t \in [0,T]. \tag{31}$$

Let z_5 denote the right hand side of the inequality (31), then z_5 is a positive and nondecreasing function on $[0,T]$ with

$$z_4(t) \le -\ln(-z_5(t)), t \in [0,T], \tag{32}$$

$$z_5(0) = -\exp(-z_4(0)) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds. \tag{33}$$

Using (32) we have

$$D_t^\alpha(z_5(t)) = h(t)\frac{w(\exp(z_4(t)))}{\exp(2z_4(t))} \le h(t)\frac{w(\exp(-\ln(-z_5(t))))}{\exp(-2\ln(-z_5(t)))}, t \in [0,T]. \tag{34}$$

Using the definition of W and the rule (9), from (34) we get

$$D_t^\alpha(W(z_5(t))) = \frac{\exp(-2\ln(-z_5(t)))}{w\exp(-\ln(-z_5(t)))}D_t^\alpha(z_5(t)) \le h(t), t \in [0,T]. \tag{35}$$

From (35) we have

$$W(z_5(t)) \le W_3(z_5(0)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds, t \in [0,T]. \tag{36}$$

From (18), (22), (27), and (32), we get

$$u(t) = z_1(t)f(t) \le z_2(t)f(t) \le z_3(t)f(t) \le \exp(z_4(t)f(t)) \le \exp(-\ln(-z_5(t)))f(t). \tag{37}$$

From (19), (23), (28), and (33), we have

$$u(t) \le \exp\left\{-\ln[-W^{-1}(W(z_5(0)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds)]\right\} f(t)$$

$$\le \exp\left\{-\ln[-W^{-1}(W(-\exp(-\ln(1+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} q(s)ds) - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds)\right.$$

$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds)]\right\} f(t), t \in [0,T]. \tag{38}$$

Because $T \in [0, T_1]$ is chosen arbitrarily, we obtain the required estimation (13).The proof is completed.

Summary

In this paper, the upper bound of the embedded unknown function in the inequality is estimated explicitly by adopting novel analytical techniques

$$u(t) \le \exp\left\{ -\ln[-W^{-1}(W(-\exp(-\ln(1+\frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\alpha-1} q(s)ds) - \frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\alpha-1} g(s)ds) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds)] \right\} f(t), t \in [0, T_1].$$

Acknowledgement

This research was supported by Guangxi Natural Science Foundation (Project No. 2012GXNS FAA053009) and Scientific Research Foundation of the Education Department of Guangxi Autonomous Region of China (No. KY2015ZD103), and the high school specialty and curriculum integration project of Guangxi Zhuang Autonomous Region (No.GXTSZY2220).

References

[1] T.H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann Math. 20 (1919)292-296.

[2] R. Bellman, The stability of solutions of linear differential equations, Duke Math J. 10 (1943) 643-647.

[3] A. Abdeldaim and M. Yakout, On some new integral inequalities of Gronwall-Bellman-Pachpatte type, Appl Math Comput. 217(2011)7887-7899.

[4] H. El-Owaidy, A. Abdeldaim and A. A. El-Deeb, On some new retarded nonlinear integral inequalities and theirs Applications, Mathematical Sciences Letters, 3(3)(2014)157-164.

[5] B. Zheng, Explicit bounds derived by some new inequalities and applications in fractional integral equations, Journal of Inequalities and Applications, 2014(4)(2014)1-12.

[6] B. G.Pachpatte, Inequalities for Differential and Integral Equations, London, Academic Press, 1998.

[7] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Comput. Math. Appl. 51(2006)1367-1376.

[8] G. Jumarie, Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions, Appl. Math. Lett. 22 (2009)378-385.

[9] GC . Wu, and EWM. Lee, Fractional variational iteration method and its application. Phys. Lett. A 374, (2010)2506-2509.

[10] B. Zheng, (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. Commun. Theor. Phys. 58(2012)623-630.