Constructing and Solving Uncertain AHP Model

Wang Tiane, Feng Yicheng, Liu Qi

Information Technology College, Jilin Agricultural University, Changchun, 130118, China

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Abstract. The most consistent approximation principle of the weight of is put forward on the basis of the interval probability hypothesis, and then gives the binarization processing of uncertain judgment matrix with consistency approximation method. To make full use of the information on the basis of expert judgment matrix, the optimal weights of the relative decision-making criterion goal is given.

Introduction

Since 1977, operational research Saaty of the United States house with analytic hierarchy process , it have got the attention of many scholars from all over the world for its advantages of itself, he made it get long-term development. We know, if the traditional AHP judgment matrix elements are accurate number, this requires decision makers understand determine scale and judgment element. There are factors in the actual problem. Uncertainty and relative importance between the rule of the same level are often difficult to accurately, and then it produce the uncertain AHP. About the uncertain AHP, many scholars solved this work. Paper [1] is calculated using sampling party in article, the workload is big and not practical. Given in paper [2, 3] the consistency approximation formula is relatively simple, but due to lack of discussion of interval number judgment matrix of each interval number. In the face of interval weight for consistency approximation and after the error analysis, we rarely not choose a relatively optimal weighting. In this paper, we give the method that is better to solve this problem.

Preliminaries

AHP is a kind of effective method which is quantifies and qualitative problem, with its applicability, simplicity, practicality. With the help of a person's knowledge, experience and wisdom it who can't use classical mathematical method deal with the qualitative problems in accordance with the actual satisfaction answer. Using AHP to solve the problem, generally it can be divided into the following four steps: one the establishment of a recursive class hierarchy; Construct two judgment matrix in two; Three by the judgment matrix, calculate the relative weight of elements that are being compared; Four calculate combination weighting of every layer element relative decision-making target . These contents in article [1] are discussed in detail.

When we are judging, we need to give a method of qualitative judgment and quantitative value, namely scale. How to select scale, Saaty gave 1-9 scale, but this is not the appropriate. We can see from below the consistency in the definition of the importance of both after quantification, in fact, it is a kind of ratio, rather than the difference between the two values, i.e. different levels of value is a multiple relationships rather than a bad relationship. This is "the real problem with the mathematical structure should be consistent" the embodiment of this principle. Therefore, this article uses the [5] in the 9/9-9/1 scale. The results of two scales contrast such as table 1

RANK	1	3	5	7	9
DEGREE	same	slightly	obviously	strongly	extreme
1—9	1	3	5	7	9
9/9—9/1	9/9	9/7	9/5	9/3	9/1

Definition 1 set R, real number domain; and called closed interval $x = [x_1, x_2]$interval number, the $x1, x2 \in R, 0 < x_1 \leq x_2$ Slightly obviously strongly extreme.

When$x_1 = x_2$, interval number is usually positive number; Said the two interval Numbers $a = [a_1, a_2], b = [b_1, b_2]$as an equal, if and only if $a_1 = b_1, a_2 = b_2$, described as $a = b$.

The following definition of interval number operation:

1)$a + b = [a_1 + b_1, a_2 + b_2]$;

2)$ab = [a_1 b_1, a_2 b_2]$;

3) $a/b = [a_1/b_2, a_2/b_1]$;

Definition 2 Let $A = (A_{ij})_{n\times n}$, $A_{ij}=[a_{ij}, b_{ij}]$ for interval number judgment matrix, if it satisfies:

1) A_{ii}=[1,1],i=1,2,…,n.

2) i, j, $A_{ij}$for interval number and meet $1/9 \le a_{ij} \le b_{ij} \le 9$.

3)$a_{ij} b_{ji} = 1$,$b_{ij} a_{ji} = 1$

Definition 3 set $A = (A_{ij})_{n\times n}$ for the interval number judgment matrix, according to the $C = (c_{ij})_{n\times n}$ for A median of digital matrix referred to as the matrix, if$c_{ij} = (a_{ij}+b_{ij})/2$.

Definition 4 set $A = (A_{ij})_{n\times n}$ is an interval number judgment matrix, called A consistent interval matrix, if for any $i, j, k = 1, 2, \dots, n$,

$$A_{ij} A_{jk} = A_{ik} \quad (2.1)$$

Called (2.1) for consistency conditions.

Definition 5 set $A = (a_{ij})_{n\times n}$is a digital judgement matrix, called $W = (w_j)$ sorting vectors, if meet: $AW = \lambda_{max} W$

The λ_{max} is A largest eigenvectors

Definition 6 B=$\{A\}_m^{\infty}$ is composed of a column matrix olumns, including $A_m = (a_{ij}^{(m)})_{n\times n}$,said A=$(a_{ij})_{n\times n}$ is the limit of B, if $\forall i = 1,2, \dots, n$, $\lim(a_{ij}^{(m)}) = aij$.

Theorem 1 Digital judge that matrix $A = (a_{ij})_{n\times n}$is a necessary and sufficient condition of consistency matrix is that a row vector (or column) is proportional to the corresponding. For arbitrary $l, h = 1, 2,$ and n, $a_{li}/a_{hi} = C$(or $a_{il}/a_{ih} = C$) $i = 1,2, \dots, n$

Proof. For A is consistency matrix, so for any of the $I, j, k = 1, 2, \dots, n$, $aij = aik/ajk$ to arbitrary boundary the I, j, $a_{ik}/a_{jk} = a_{ij} = C$.

For arbitrary $l, h = 1, 2, \dots, n, a_{li}/a_{hi} = C$, $i = 1,2, \dots, n$, $a_{jj} = 1$, $j = 1,2, \dots, n$, thereby $C = a_{lh}$, so for any $l, h = 1, 2, \dots, n$, $a_{lh} = \frac{a_{li}}{a_{hi}}$, $i = 1,2, \dots, n$.. Therefore A is consistency matrix.

Theorem 2 Digital judge matrix $A = (a_{ij})$ that A is necessary and sufficient condition of consistency matrix is$\forall i, j \in \mathbf{\Omega}$, its ranking vector satisfy:

$$a_{ij} = w_i/w_j \quad (2.2)$$

Basic Assumptions and Principles

(1) suppose the interval probability assumption ;importance of the two criteria in the corresponding interval probability values are 1, and it obedient on interval number as the interval numbers the values of normal distribution.

When we are judging, there is the characteristic, namely in the subconscious we will put that expect to achieve quantitative value judgment on the median location of interval numbers. And we know that normal distribution is the most common form of distribution in the nature. Thus,we make the above hypothesis.

(2) the principle of consistency approximation.

Principle 1: consistency approximation matrix may be obtained, which is completely consistent.

Principle 2: we make approximation matrix close to the median matrix, and the matrix approximation value matrix in relative deviation is as small as possible.

Principle 3: approximation matrix elements should be close to the corresponding interval numbers in the interval number judgment matrix.

Before this principle, first I make a point: the experts make judgment after considered, after we get the interval number judgment matrix, the decision-makers decision-making nearly all the information in it. We make full use of the remaining work is to construct mathematical model of interval number judgment matrix which is given information to give a satisfactory decision.

Principle 1 is a required by the problem itself to do, otherwise the approximation is meaningless. Principles2, 3 on the interval probability hypothesis is also should meet .The following work is to seek a kind of mathematical model to make it meet the three principles.

By interval probability hypothesis, we known interval number near the median value is what we want to close to. So, we call the median matrix of the interval number judgment matrix the uncertain AHP binarization processing.

Uncertain Consistency Approximation and the Generalized Least Square Method

Set number judgment matrix, the ranking vector for $W = (w_1, \ldots, w_n)^T$ and satisfy the normalization conditions

$$\sum w_i = 1 \tag{5.1}$$

Remember vector space $\mathbf{D} = \{W = (w_1, w_2, , w_n)^T \mid w_j > 0, j = 1,2, \ldots, n, w_j = 1\}$ collection $\boldsymbol{\omega} = \{1,2, \ldots, n\}$,

When A is consistency judgment matrix, according to the theorem 2.2 There are

$$a_{ij} = w_i/w_j \quad i, j \in \boldsymbol{\omega} \tag{5.2}$$

$(w_i/w_j)/w_{ij} - 1 = 0$, that is $(w_i/w_j - w_{ij})/w_{ij} = 0$

Remember the delta $\delta_{ij} = (w_i/w_j - a_{ij})/ a_{ij}$ called relative deviation.

$$\text{The delta } \delta_{ij} = 0 \quad \text{, any} i, j \in \boldsymbol{\omega} \tag{5.3}$$

Obviously, because of a consistent condition (2.1 ') and the expert knowledge structure, level of judgment for preference and vagueness of the objective things themselves are often difficult to meet, especially when the order number judgment matrix are higher, thus (5. 3) is usually wrong. To this end, we introduce the general deviation f_{ji},, order

$$f_{ij} = \delta_{ij},^2 \quad i, j \in \boldsymbol{\Omega}$$

At the same time, generalized bias function for construction

$$F(W) = \sum\sum f_{ij} \quad i, j \in \boldsymbol{\omega}$$

Obviously, in order to meet the principle of one, two, it is clearly the value of F (W) as small as possible, so we can find a mathematical model of the problem:

$$W^*:$$

$$F(W^*) \leq F(W) \quad \forall W \in \mathbf{D}$$

In fact, we also not need use square to construct the generalized variation function, and with a higher power, such as four, eight, etc. High power makes the relative deviation between the deviation is smaller. We don't need high power that is given to solve the difficulty, therefore, using the low power of square. But with the back of the correction, we still can achieve satisfactory answer.

Theorem 5.1 generalized deviation function F in D space has the minimum point W $*$, and W $*$ is a system of equations

$$\sum[(a_{ij}w_j/w_i)^2 - a_{ij}w_j/w_j] = \sum[(a_{ji}w_i/w_j)^2 - a_{ji}w_i/w_j] \quad i \in \boldsymbol{\omega} \tag{5.4}$$

Theorem 5.2 Set F (w) is a matrix $A = (a_{ij})_{n\times n}$ corresponding to the generalized deviation function, w $*$ is F (w) the minimum point of the space in D. The necessary and sufficient condition of A consistency matrix is F $*$ (w) = 0.

Proof. ∵ A is consistency matrix, according to the theorem 2. 2, know

$$\forall i, j \in \boldsymbol{\Omega}, a_{ij} = w_i/w_j$$

Make W0 is sort of vector A,

Apparently $F(w_0) = 0$,

And ∵ $F(w) \geq 0$, $\forall w \in \mathbf{D}$

W_0 is F (W) in D minimum point in the space.

$\therefore\ W^* = W_0$

Thus, F（W^*） $= 0$.

$\because F(W^*) = 0$ is defined by generalized deviation function

$$\delta_{ij} = 0$$

$\therefore\ a_{ij} = W^*{}_i / W^*{}_j$, according to the theorem 2. 2,

A is consistency matrix.

An iterative convergence of algorithm Equations (5.4) is given below.

(1) we choose the initial sorting fixed vector$W(0) = (w_1(0), w_2(0), \dots w_n(0))T \in \mathbf{D}$, and known epsilon iteration precisionε, at the same time set k = 0.

(2)calculate $i(w(k)) = \{[(a_{ij}w_j/w_i)^2 - (a_{ij}w_j/w_i) - [(a_{ji}w_i/w_j)^2 - a_{ji}w_i/w_j] = \{[(a_{ij}w_j/w_i)^2 - (a_{ji}w_i/w_j)^2] - [a_{ij}w_j/w_i - a_{ji}w_i/w_j]\}$

If$\forall i \in \mathbf{\Omega}$, constant have $|\eta_i(w(k)| < \varepsilon$, termination, or (3).

(3) determine m, make $|\eta_i(w(k))| = \max|\eta_i(w(k))|$, and calculate the$\Theta = [(a_{mj}w_j/w_m)^2 - (a_{jm}w_m/w_j)^2]$

If$\Theta \geq \varepsilon/2$ (or else turn (4)), then

$$\begin{cases} T(k) = \{[\sum(a_{mj}w_j/w_m)^2]/[\sum a_{jm}w_m/w_j)^2]\}^{1/4} \\ x_i(k) = \begin{cases} T(k)w_m(k), i = m \\ w_i(k), i \neq m \end{cases} \end{cases}$$

$$w_i(k+1) = x_i(k)/x_j(k) \quad i \in \boldsymbol{\omega}$$

$$k = k + 1, \quad \text{turn (2)}$$

$K = k + 1$, turn (2)

(4) when the < / 2, it is

$T(k) = [\sum(a\_mj\ w\_j/w\_m)/\sum(a\_jm\ w\_m/w\_j)]^{1/2}$

$$X_i(k) \begin{cases} x_i(k) = T(k)w_m(k) & i = m: \\ w_i(k) & i \neq m \end{cases}$$

$$w_i(k+1) = x_i(k)/x_j(k) \qquad i \in \mathbf{\Omega}$$

$K = k + 1$,turn (2)

Theorem 5.3 set $A = (a_{ij})_{n\times n}$ for the digital judgement matrix, if $\exists l, h \in \mathbf{\Omega}$ and $a_{li}/a_{hi} = C, i \in \mathbf{\Omega}$，C to A certain value, W $*$ obtained from GLSM method satisfies: $w_l/w_h = a_{lh}$.

Examples

To comprehensive evaluate safety performance of a concrete continuous girder bridge, a method introduced in this paper is used to obtain the weight of evaluation index. Here we only gives a concrete quality evaluation model, consider the following five indicators: the apparent quality of concrete crack, protective layer thickness, steel corrosion and concrete strength. By expert scoring and comprehensive, the interval number judgment matrix of evaluation index is given.

$$\begin{pmatrix} [1,1] & [0.5,1] & [3,5] & [0.4,0.67] & [0.25,0.33] \\ & [1,1] & [4,5] & [0.5,1] & [0.29,0.4] \\ & & [1,1] & [0.2,0.25] & [0.14,0.17] \\ & & & [1,1] & [0.5,1] \\ & & & & [1,1] \end{pmatrix}.$$

To calculate the optimal weight is (0.15, 0.18, 0.15, 0.18, 0.43).

Conclusion

In this paper for the determine of uncertain AHP judgment matrix to the weighing values of the optimal, we put forward the corresponding principle and algorithm. Two practical calculations show that the method is effective. But when the proposed algorithm to calculate sometimes is not stable, the next step of work is to solve this problem.

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