New existence criteria for periodic solution to a Duffing p-Laplacian-Like equation

Zhiyan Li[1,a]

[1]Department of Mathematics and Physics, Hohai University, Changzhou Campus, Changzhou, P.R. of China

[a]lizhiyan_hh@163.com

Keywords: p-Laplacian-Like operator, Periodic solution, Degree theory, Duffing equation

Abstract. In this study, we investigate a kind of Duffing type p-Laplacian-Like equation. Some new criteria for guaranteeing the existence and uniqueness of periodic solution of this equation are given by using the Manásevich-Mawhin continuation theorem and some analysis techniques. Our results improve and extend some known results from the literature.

Introduction

In this paper, we consider the existence and uniqueness of periodic solution for the following nonlinear equation with p-Laplacian-Like operator

$$(\phi(x'(t)))'+Cx'(t)+g(t,x(t))=e(t), \quad (1)$$

where g is continuous and differential on R^2, and e is a continuous function on R with period $T>0$, C is a given constant; moreover, ϕ is called p-Laplacian-Like operator satisfying the following conditions:

(H_1) $\forall$ $x_1,x_2\in R, x_1\neq x_2, [\phi(x_1)-\phi(x_2)]\cdot(x_1-x_2)>0$, ϕ is continuous and $\phi(0)=0$;

(H_2) There exists a function $\alpha:[0,+\infty)\to[0,+\infty)$ such that $\lim_{s\to+\infty}\alpha(s)=+\infty$ and

$$\phi(x)\cdot x\geq\alpha(|x|)|x|^2, \quad \forall x\in R.$$

It's obvious that under conditions (H_1) and (H_2), ϕ is an isomorphism from R onto R and is increasing on R. As is known, p-Laplacian operator $\varphi_p(s)=|s|^{p-2}s, p>1$, it is a special case of ϕ. However, there are few results about the existence of periodic solutions to Eq. (1), the difficulty lies in that ϕ is more complicated than p-Laplacian operator and ϕ has no concrete form. There are many results about p-Laplacian operator, see examples [1-7].

Recently, Wang [8] considered the following p-Laplacian equation

$$(\varphi_p(x'(t)))'+Cx'(t)+g(t,x(t))=e(t). \quad (2)$$

Under some conditions, they have obtained the existence and uniqueness.

Remark. When $\phi=\varphi_p(s)$, Eq. (1) is Eq. (2), that is to say Eq. (2) is a special case of Eq. (1).

Our main results read as follows:

Theorem 1. In the problem (1), assume that

(A_1) There exists a constant $d>0$ such that $xg(t,x)<0, \forall t\in R$ and $|x|\geq d$,

(A_2) $[g(t,u_1)-g(t,u_2)]\cdot(u_1-u_2)<0, u_1\neq u_2$ hold.Then Eq. (1) has an unique periodic solution.

Some lemmas and notationsThe following lemma 1 is necessary for the proof of Theorem1.

Lemma 1 [9]. Let (H_1), (H_2) hold and $\tilde{f}$ is Carathéodory. Assume that Ω is an open bounded set in C_T^1 such that the following conditions are satisfied:

(S_1) For each $\lambda\in(0,1)$, the problem $(\phi(x'(t)))'=\lambda\tilde{f}(t,x,x'), x(0)=x(T), x'(0)=x'(T)$ has no solution on $\partial\Omega$,

(S_2) The equation $F(a)=\frac{1}{T}\int_0^T \tilde{f}(t,a,0)dt=0$ has no solution on $\partial\Omega\cap R$.

(S_3) The Brower degree deg $(F,\Omega\cap R,0)\neq 0.$ Then the periodic boundary value problem $(\phi(x'(t)))'=\tilde{f}(t,x,x'), x(0)=x(T), x'(0)=x'(T)$ has at least one T-periodic solution on $\overline{\Omega}$.The following notations will be used throughout the rest of this study.

$$|x|_\infty=\max_{t\in[0,T]}|x(t)|,\quad |x'|_\infty=\max_{t\in[0,T]}|x'(t)|,\quad |x|_k=\left(\int_0^T|x(t)|^k\,dt\right)^{\frac{1}{k}},$$

$$C_T^1:=\{x\in C^1(R,R): x(t+T)=x(T)\},\ C_T:=\{x\in C(R,R): x(t+T)=x(T)\}.$$

C_T, C_T^1 are two Banach spaces with the norms $\|x\|_{C_T^1}=\max\{|x|_\infty,|x'|_\infty\}, \|x\|_{C_T}=|x|_\infty$.

The proof of theorem 1

We are now in the position to give the proofs of Theorem 1.Proof. We will prove theorem in two steps.Existence.

We will show that Eq. (1) has at least one T-periodic solution. Consider the homotopic equation of Eq. (1):

$$(\phi(x'(t)))'+\lambda Cx'(t)+\lambda g(t,x(t))=\lambda e(t),\ \lambda\in(0,1), \qquad (3)$$

First, we prove that the set of the T-periodic solution to Eq. (3) is bounded in C_T^1. Let $S\subset C_T^1$ be the set of T-periodic solution of Eq. (3). If $S=\Phi$, the proof is ended. Suppose $S\neq\Phi$ and $x\in S$. Noticing that $x(0)=x(T), x'(0)=x'(T)$, $\phi(0)=0$ and $\int_0^T e(t)dt=0.$ Integrating from 0 to T of Eq. (3), we have $\int_0^T g(t,x(t))dt=0$, which implies that there exists a point $t_0\in[0,T]$ such that

$$g(t_0,x(t_0))=0. \qquad (4)$$

By (A_1) and (4), we have $|x(t_0)|<d$.

So $|x(t)|<d+\int_0^T|x'(s)|ds.$ That is $|x|_\infty<d+|x'|_1$. (5)

In view of $\lim_{s\to+\infty}\alpha(s)=+\infty$, there is a constant $K>0$ such that $\alpha(s)\geq 1,\forall s\geq K$.Define $E_1=\{t: t\in[0,1],|x'(t)|>K\},\ E_2=\{t: t\in[0,1],|x'(t)|<K\}$, $E_3=\{t: t\in[0,1],|x(t)|>d\},\ E_4=\{t: t\in[0,1],|x(t)|<d\}$. By ($H_2$) and (5), we obtain

$$\begin{aligned}
\int_0^T|x'(t)|^2dt &= \int_{E_1}|x'(t)|^2dt+\int_{E_2}|x'(t)|^2\,dt\\
&\leq \int_{E_1}|x'(t)|^2dt+K^2T\\
&\leq \int_{E_1}\frac{\phi(x'(t))\cdot x'(t)}{\alpha(|x'(t)|)}dt+K^2T\\
&\leq \int_{E_1}\phi(x'(t))x'(t)dt+K^2T\\
&\leq \int_0^T\phi(x'(t))x'(t)dt+K^2T\\
&= -\int_0^T(\phi(x'(t)))'x(t)dt+K^2T\\
&= \lambda\int_0^T g(t,x(t))\cdot x(t)dt-\lambda\int_0^T e(t)x(t)dt+K^2T
\end{aligned}$$

$= \lambda\int_{E_3} g(t,x(t))\cdot x(t)dt + \lambda\int_{E_4} g(t,x(t))\cdot x(t)dt$

$-\lambda\int_0^T e(t)x(t)dt + K^2T$

$\le \lambda\int_{E_4} g(t,x(t))\cdot x(t)dt - \lambda\int_0^T e(t)x(t)dt + K^2T$

$\le \int_{E_4} |g(t,x(t))|\cdot|x(t)|dt + \int_0^T |e(t)||x(t)|dt + K^2T$

$\le (\max_{t\in[0,1],|x|\le d} |g(t,x)| + |e|_\infty)T|x|_\infty + K^2T$.

Let

$M_0 = (\max_{t\in[0,1],|x|\le d} |g(t,x)| + |e|_\infty)T$.

Then we get

$$|x'|_2^2 \le M_0|x|_\infty + K^2T. \qquad (6)$$

By (4), (6) and Holder inequality, we have

$$|x'|_1 \le \sqrt{T}(M_0|x|_\infty + K^2T)^{\frac{1}{2}} \le \sqrt{T}(M_0(d+|x'|_1) + K^2T)^{\frac{1}{2}}$$

which yields that there exist a constant $M_1 > 0$ such that

$$|x'|_1 < M_1, |x|_\infty < d + M_1. \qquad (7)$$

Since $x(0) = x(T)$, there must exist t_0^* such that $x'(t_0^*) = 0$, from Eq. (1) we get

$|\phi(x'(t))| = |\int_{t_0^*}^t (\phi(x'))'ds|$

$= \lambda|\int_{t_0^*}^t (Cx'(s) + g(s,x(s)) - e(s))ds|$

$\le \int_0^T (C|x'(s)| + |g(s,x(s)| + |e(s)|)ds$

$< CM_1 + (G + |e|_\infty)T,$

where $G = \max\{|g(t,x) : t\in[0,1], |x| \le d + M_1\}$, so we have

$$|x'|_\infty < \phi^{-1}(CM_1 + (G + |e|_\infty)T). \qquad (8)$$

Let $M = \max\{d + M_1, \phi^{-1}(CM_1 + (G + |e|_\infty)T)\}$,

then from (7), (8),

$\|x\| < M$.

This means condition (S_1) of Lemma 1 is satisfied. Next we show condition (S_2), (S_3) are also satisfied.

Set

$\tilde{f}(t,x(t),x'(t)) = -Cx'(t) - g(t,x(t)) + e(t)$,

thus

$(\phi(x'(t)))' = \lambda\tilde{f}(t,x(t),x'(t))$.

Denote $B = \{x : x\in C_T^1, \|x\| < r, r \ge M\}$, by (S_2)

$$F(a) = \frac{1}{T}\int_0^T \tilde{f}(t,a,0)dt = \frac{1}{T}\int_0^T (e(t) - g(t,a))dt$$

$$= -\frac{1}{T}\int_0^T g(t,a)dt, \qquad (9)$$

this together with (A_1), we get

$F(r)F(-r) < 0$.

For $x \in \partial\Omega \cap R, \mu \in [0,1]$,

$$xH(x,\mu) = \mu x^2 - (1-\mu)x \cdot \frac{1}{T}\int_0^T g(t,x)dt > 0,$$

we can see

$\deg(F, \Omega \cap R, 0\} = \deg(I, \Omega \cap R, 0) \neq 0.$

This means (S_2), (S_3) of Lemma 1 hold. By applying Lemma 1 , there exists at least one solution with periodic T to Eq. (1). The existence is now completed.Uniqueness Let

$\psi(x) = \int_0^x Cdu = Cx,\ y(t) = \phi(x'(t)) + \psi(x(t)).$ Then Eq. (1) is transformed into

$$x'(t) = \phi^{-1}[y(t) - \psi(x(t)],$$
$$y'(t) = -g(t, x(t)) + e(t).$$

Let $x_1(t)$ and $x_2(t)$ be two T-periodic solution of Eq. (1), and

$y_i(t) = \phi(x_i'(t)) + \psi(x_i(t)),\ i = 1,2.$ Then we obtain

$$x'_i(t) = \phi^{-1}[y_i(t) - \psi(x_i(t)],$$
$$y_i'(t) = -g(t, x_i(t)) + e(t).$$

Setting

$$v(t) = x_1(t) - x_2(t),\ u(t) = y_1(t) - y_2(t), \tag{10}$$

it follows from (10) that

$$v'(t) = \phi^{-1}[y_1(t) - \psi(x_1(t))] - \phi^{-1}[y_2(t) - \psi(x_2(t))]$$
$$u'(t) = -[g(t, x_1(t)) - g(t, x_2(t))].$$

Now, we claim that $u(t) \le 0,\ \forall t \in R.$

We argue by contradiction . Suppose there exists $t_2 \in (0,T)$ such that $u(t_2) = \max_{t\in[0,T]} u(t)$,which implies that

$$u'(t_2) = -[g(t_2, x_1(t_2)) - g(t_2, x_2(t_2))] = 0.$$
$$u''(t_2) = -[g_1'(t_2, x_1(t_2)) + g_2'(t_2, x_1(t_2))x_1'(t_2) - g_1'(t_2, x_2(t_2)) - g_2'(t_2, x_2(t))x_2'(t_2)] \le 0,$$

where $g_1' = \frac{\partial g(t,x)}{\partial t}$, $g_2' = \frac{\partial g(t,x)}{\partial x}$.

By (A_2), $g_2'(t,x) < 0.$ So it follows from $g(t_2, x_1(t_2)) - g(t_2, x_2(t_2)) = 0$ that $x_1(t_2) = x_2(t_2).$

Thus, in view of $-g_2'(t_2, x_1(t_2)) > 0, u(t_2) = y_1(t_2) - y_2(t_2) > 0,$

Together with (A_2), we obtain $u''(t_2) = -[g_2'(t_2, x_1(t_2))x_1'(t_2) - g_2'(t_2, x_2(t))x_2'(t_2)]$

$$= -g_2'(t_2, x_1(t_2))\{\phi^{-1}[y_1(t_2) - \psi(x_1(t_2))] - \phi^{-1}[y_2(t_2) - \psi(x_1(t_2))]\} > 0,$$

which contradicts with $u''(t) \le 0.$ Now , we have proved that $u(t) \le 0, \forall t \in R.$

Analogously, one can show that $u(t) \ge 0, \forall t \in R.$

So we have $u(t) \equiv 0$. That is $x_1(t) = x_2(t), \forall t \in R$

Hence , Eq. (1) has an unique T-periodic solution. The proof of Theorem 1 is now completed

References

[1] X.Q. Liu, Y.X. Guo and J.Q. Liu, Solutions for singular p-Laplacian equation in R^N, Jrl Syst Sci & Complexity (2009) 22: 597-613.

[2] S.P. Lu, Z.J. Gui, On the existence of periodic solutions to p-Laplacian Rayleigh differential equation with a delay, J. Math. Anal. Appl. 325(2007)685-702.

[3] W.S. Cheung, J.L. Ren, Periodic solutions for p-Laplacian Liénard equation with a deviating argument, Nonlinear Anal. 59(2004)107-120.

[4] S.P. Lu, W.G. Ge, Periodic solutions for second order p-Laplacian differential equation with a deviating argument (in Chinese), Acta. Math. Sin. (2005) 48 (5)841-850.

[5] S.P. Lu, W.G. Ge, Periodic solutions for a kind of Liénard equations with deviating arguments. J. Math. Anal. Appl. 249 (2004)231-243.