Research on the double slanted cantilevers of a micro structure based on Materials Mechanics

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Abstract. The double slanted cantilevers are used in some micro-structure designs, which have some unique characters. In this paper, the mathematical model of the double slanted cantilevers with irregular cross-section was investigated based on the theory of material mechanics. The moment of inertia and principal axes of the cross-section was derived on some hypothesis. The displacement sensitivity was also analyzed. And FEM simulations validate the calculations.

Introduction

The cantilever structure is widely used in micro-structure sensors [1], such as the accelerometer and the gyroscope used in inertia navigation system [2], and micro probe used in biochemistry detection [3]. The mechanical character of the cantilever has a close relationship with its dimensions, as well as its moment of inertia and Young's modulus, etc.

The double slanted cantilever has a good performance in a mechanical sensor [4]. But there is no detailed mathematical analysis for the double cantilevers structure.

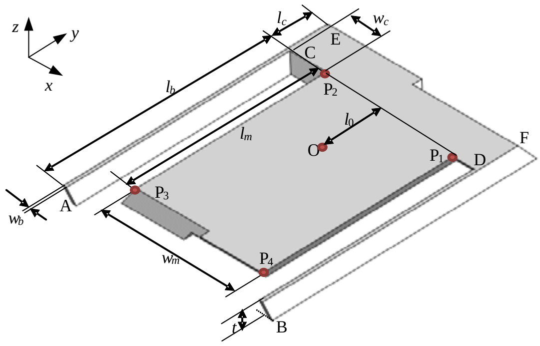
This paper discussed the mechanical characteristics of a kind of double slanted cantilevers based on a mathematical model. The model grounds on the theory of mechanics of materials.

For the cantilevers researched in, there is a mass block suspended on the end. And the cantilevers are slanted to the same direction with a specific angle.

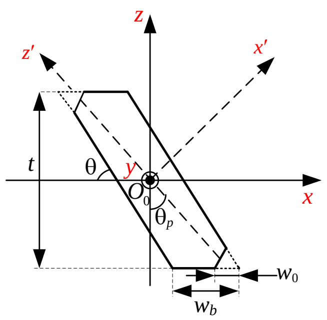
Modeling of the double slanted cantilevers

The model of the double slanted cantilevers with a block is established as following.

As is shown in Figure 1 (a), the big mass block, $CEP_2P_3P_4P_1DF$, which can be seen as a rigid body, is suspended on the two slanted cantilevers at point C and point D. While the two slanted cantilevers, AC and BD, which are flexible, are fixed to a base at point A and point B. The point O is the centroid of the mass block. And the cross-section is shown in Figure 1 (b), which is an irregular hexagon, equivalent to a parallelogram cut off two identical isosceles triangles at the sharp angles.



(a) Geometry of the cantilever with a block



(b) the cross-section of the cantilever

Fig. 1 The draft of the slanted cantilever

The meanings of the symbols are listed on the following table.

Table 1 the meanings of the symbols in Figure 1

w_b	Width of one cantilever	l_b	Length of on cantilever
w_c	Width of the front end of the block	l_c	Length of the front end of the block
w_m	Width of the block	l_m	Length of the block
l_0	Distance from the mass center to the end of the cantilever	t	Thickness of the block
θ	The slanted angle		

As the forces on the two cantilevers are symmetrical, analysis of one cantilever is enough. As is shown in Figure 1(b), we set up two coordinate systems on the cross-section of the cantilever. The x-axis and z-axis are separately parallel to the axes in Figure 1 (a). The z'-axis as well as the x'-axis is the principle axis of the irregular cross-section.

According to the formula of geometry [5], we get

$$l_0 = \frac{l_m l_m w_m - l_c(w_m + 2w_c)l_c}{2(l_m w_m + l_c(w_m 2w_c))} \tag{1}$$

We get the moment of inertia of the cross-section by the x-axis by integration [5].

$$\begin{aligned} I_x &= \frac{w_b t^3}{12} - \int_{-\frac{t}{2}}^{-\frac{t}{2}+\frac{w_0\tan\theta}{2}} \frac{z^2}{\tan\theta}(-t + w_0\tan\theta - 2z)\mathrm{d}z - \int_{\frac{t}{2}-\frac{w_0\tan\theta}{2}}^{\frac{t}{2}} \frac{z^2}{\tan\theta}(-t + w_0\tan\theta + 2z)\mathrm{d}z \\ &= \frac{1}{48}(4t^3 w_b - 6t^2 w_0^2\tan\theta + 4t w_0^3\tan^2\theta - w_0^4\tan^3\theta) \end{aligned} \tag{2}$$

In the same way, we get the moment of inertia of the cross-section by the y-axis

$$I_z = \frac{1}{48}(4t(w_b^3 - 3w_b w_0^2 + 3w_0^3) - 6t^2 w_0^2\cot\theta + 4t^3 w_b\cot\theta^2 - w_0^2(6w_b^2 - 12w_b w_0 + 7w_0^2)\tan\theta) \tag{3}$$

And the product of inertia

$$I_{xz} = \frac{1}{24}(t - w_0\tan\theta)(2t^2 w_b\cot\theta + w_0(t(2w_b - 3w_0) - (w_b - w_0)w_0\tan\theta)) \tag{4}$$

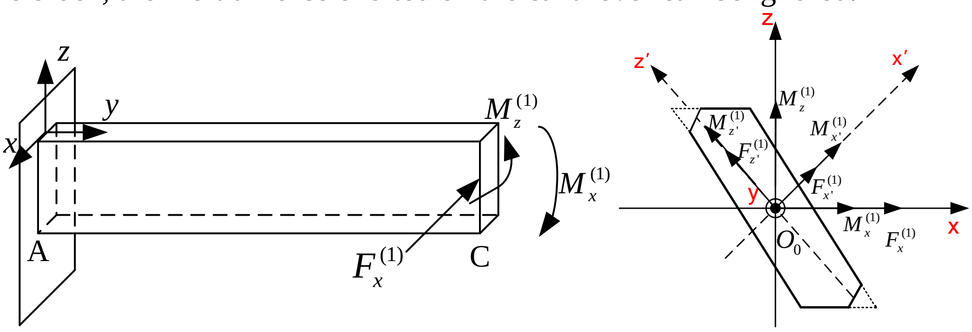
The principle axis angle θ_0 can be obtained by solving the following equation

$$\tan\theta_0 = \tan(\frac{\pi}{2} - \theta_p) = \frac{2I_{xz}}{I_x - I_z} \tag{5}$$

The principle moment of inertia of the cross-section by the x' and z' axis can be obtained.

$$\begin{cases} I_{x'} = \dfrac{I_z + I_x}{2} - \dfrac{(I_z - I_x)\cos 2\theta_p}{2} + I_{xz}\sin 2\theta_p \\ I_{z'} = \dfrac{I_z + I_x}{2} + \dfrac{(I_z - I_x)\cos 2\theta_p}{2} - I_{xz}\sin 2\theta_p \end{cases} \tag{6}$$

When the mass block is pushed by an inertial force along the x-axis or z-axis, it will displace from its original position. The displacement of the block is determined by the movement of the point C and D on the double slanted cantilevers. Because the mass of the cantilever is very small compared to the block, the inertial force exerted on the cantilever can be ignored.



(a)The force analysis of the cantilever (b) the force analysis of the cross-section at point C

Fig. 2 Force analysis about an x-axis direction force on the cantilever

As is shown if Figure 2, the block is suspended by two cantilevers, each cantilever endures half of the force of the block. According to Newton's second law, the force exerted on point C can be obtained

$$F_{x'}^{(1)} = -\frac{m_0 a_x}{2} \tag{7}$$

In the upper equation, m_0 is the mass of the block, and a_x is the acceleration along the *x*-axis.

By decomposing the force and torque along the two principle axes, we get the following equations

$$\begin{cases} F_{x'}^{(1)} = F_x^{(1)} \cos\theta_p \\ F_{z'}^{(1)} = -F_x^{(1)} \sin\theta_p \end{cases}; \begin{cases} M_{x'}^{(1)} = M_x^{(1)} \cos\theta_p + M_z^{(1)} \sin\theta_p \\ M_{z'}^{(1)} = -M_x^{(1)} \sin\theta_p + M_z^{(1)} \cos\theta_p \end{cases} \tag{8}$$

According to the laws in mechanics of materials [6], the displacement together with the torsion angle on point C around the principle axes is

$$\begin{cases} w_{x'}^{(1)} = \dfrac{F_{x'}^{(1)} l_b^{\,3}}{3E_0 I_{z'}} - \dfrac{M_{z'}^{(1)} l_b^{\,2}}{2E_0 I_{z'}} \\ w_{z'}^{(1)} = \dfrac{F_{z'}^{(1)} l_b^{\,3}}{3E_0 I_{x'}} + \dfrac{M_{x'}^{(1)} l_b^{\,2}}{2E_0 I_{x'}} \end{cases}; \begin{cases} \theta_{x'}^{(1)} = \dfrac{F_{z'}^{(1)} l_b^{\,2}}{2E_0 I_{x'}} + \dfrac{M_{x'}^{(1)} l_b}{E_0 I_{x'}} \\ \theta_{z'}^{(1)} = -\dfrac{F_{x'}^{(1)} l_b^{\,2}}{2E_0 I_{z'}} + \dfrac{M_{z'}^{(1)} l_b}{E_0 I_{z'}} \end{cases} \tag{9}$$

In Equation 9, the symbol E_0 represents the Young's modulus.

According the coordinate transformation rules, we can translate the displacement and the angle from the coordinate system x'-O_0-z' to the coordinate system x-O_0-z [7].

$$\begin{bmatrix} w_x^{(1)} \\ w_z^{(1)} \end{bmatrix} = \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \begin{bmatrix} w_{x'}^{(1)} \\ w_{z'}^{(1)} \end{bmatrix}; \begin{bmatrix} \theta_x^{(1)} \\ \theta_z^{(1)} \end{bmatrix} = \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \begin{bmatrix} \theta_{x'}^{(1)} \\ \theta_{z'}^{(1)} \end{bmatrix} \tag{10}$$

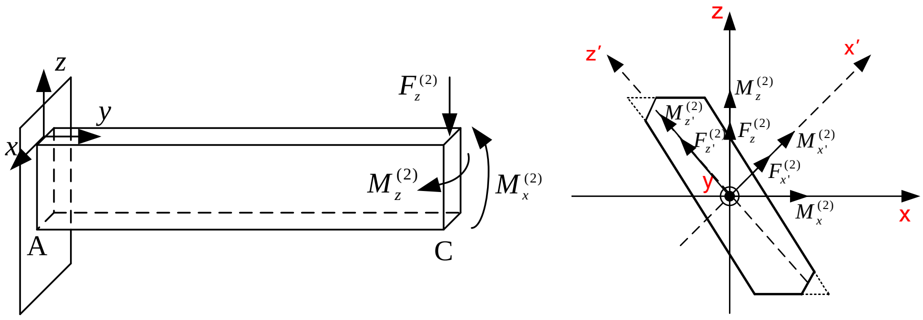
Apply boundary conditions, $\theta_x^{(1)}=0$, $\theta_z^{(1)}=0$, we have the solution of the torque

$$\begin{cases} M_x^{(1)} = 0 \\ M_z^{(1)} = -\dfrac{l_b}{4} m_0 a_x \end{cases} \tag{11}$$

Take Equation 11 into Equation 9 and 10, we get the relationship of $w_z^{(1)}$ with the *x*-axis acceleration a_x

$$w_z^{(1)} = \frac{(-I_{x'} + I_{z'}) l_b^{\,3} m_0 \cos\theta_p \sin\theta_p}{24 E_0 I_{x'} I_{z'}} a_x \tag{12}$$

Analogously, when a z-axis inertial force applied on the block, we can have the force decomposition in Figure 3.



(a)The force analysis of the cantilever (b) the force analysis of the cross-section at point C

Fig. 3 Force analysis about a *z*-axis direction force on the cantilever

Following the same derivation methods and principles, we can get relationship of the displacement along the *z*-axis at point C on the slanted cantilever with the acceleration a_z, and the boundary conditions is $\theta_z^{(2)} = 0$.

$$w_z^{(2)} = \frac{m_0 l_b^{\,2}(6I_{x'}I_{z'}(8l_0 - 5l_b) - I_{x'}^{\,2} l_b - I_{z'}^{\,2} l_b + (I_{x'} - I_{z'})^2 l_b \cos 4\theta_p)}{192 E_0 I_{x'} I_{z'} (I_{x'} \cos\theta_p^{\,2} + I_{z'} \sin\theta_p^{\,2})} a_z \tag{13}$$

At the same time, the torsion angle at point C around the x-axis is obtained

$$\theta_x^{(2)} = \frac{(2l_0 - l_b)l_b m_0}{4E_0(I_{x'}\cos\theta_p{}^2 + I_{z'}\sin\theta_p{}^2)} a_z \tag{14}$$

In the mechanical sensor mentioned above [4], there is an electric pad faced the mass block across a thin gap. The distance from the point on the block to the electric pad is written as

$$w_z(y) = d_0 + w_z(0) + y\theta_x \tag{15}$$

In Equation 15, d_0 is the original distance between the electric pad and the mass block, y is the coordinate in the y-axis of a specific point on the block, and the origin locates where the block suspended on the cantilevers. As can be seen, w_z is related to both translation and rotation.

The displacement sensitivity can be obtained by a differential equation

$$S_z = \frac{\partial w_z}{\partial a} \tag{16}$$

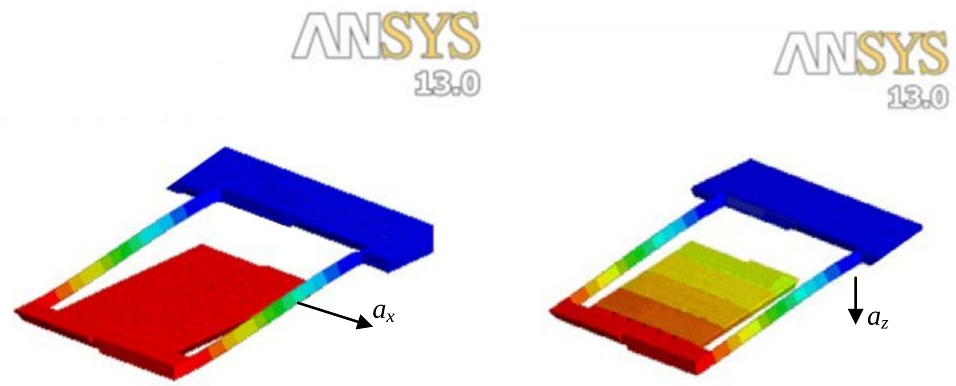
For the displacement sensitivity, it tells the relationship between the acceleration of force exerted on the block and the displacement along the z-axis. It's the basis of some sensor designs.

FEM Simulations

In order to validate the derivation results, FEM simulations were done using the parameters in the below table.

Table 2 the value used in the simulations and calculations

w_b	l_b	w_c	l_c	w_m	l_m
0.07×10^{-3}m	4.15×10^{-3}m	0.65×10^{-3}m	0.77×10^{-3}m	2.97×10^{-3}m	4.37×10^{-3}m
l_0	t	θ	E_0	v	ρ
1.34×10^{-3}m	0.24×10^{-3}m	54.74°	169×10^{9}Pa	0.3	2330 kg/m^3



(a) The x-axis acceleration result (b) the z-axis acceleration result

Fig. 4 Simulations contour plot of the structure

As is shown in Figure 4 (a), when the x-axis acceleration exerted on the structure, the block has a transition with the displacement in the z-axis in accord with Equation 12.

In Figure 4 (b), the z-axis acceleration will cause a transition as well as a rotation, as can be obtained in Equation 13 and 14.

Pick four points P1, P2, P3, and P4 on the block in Figure 1 to see the displacements. The contrast of the FEM simulations and the calculations based on the equations is shown in table 3 and 4.

Table 3 the z-axis displacement values of the points P_1, P_2, P_3, and P4 by x-axis acceleration

Displacement	$w_z^{(1)}$	
	Simulation value($\times10^{-8}$m)	Calculation value($\times10^{-8}$m)
$d_z(P_1)$	-1.26	-1.25725
$d_z(P_2)$	-1.26	-1.25725
$d_z(P_3)$	-1.27	-1.25725
$d_z(P_4)$	-1.27	-1.25725

Table 4 the z-axis displacement values of the points P_1, P_2, P_3, P_4 by z-axis acceleration

Displacement	$w_z^{(2)}$	
	Simulation value ($\times 10^{-8}$m)	Calculation value ($\times 10^{-8}$m)
$d_z(P_1)$	-1.24	-1.23152
$d_z(P_2)$	-1.23	-1.23152
$d_z(P_3)$	-0.958	-0.89949
$d_z(P_4)$	-0.945	-0.89949

From the contrast, the displacement calculation results of the equations derived in the second part based on the materials mechanics were almost the same with the FEM simulations results. Therefore, the correctness of the mathematic models is validated.

Conclusion

On the basis of the theory of mechanics of materials, the mathematical model of the double slanted cantilevers was detailed investigated. The moment of inertia of the cross-section and the displacement sensitivity of the structure were derived.

The mathematic model of the double slanted structure is important for optimization design. And how each parameter of the structure influence its mechanical character is pretty distinct by analyzing the equations obtained in the paper, which laid the foundation of deeper investigations.

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