Hybrid control of Hopf bifurcation in a fractional order small-world network model

Lina MA[1,a], Xiuli ZHANG[1,b], Dawei DING[1,c,*]

[1]School of Electronics and Information Engineering, Anhui University, Hefei, China

[a]1195464263@qq.com, [b]632358763@qq.com, [c]dwding@adu.edu.com

*Corresponding author

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Abstract. In this paper, a hybrid control strategy is applied to control the Hopf bifurcation in a fractional order small-world network model. By choosing time-delay as a bifurcation parameter and analyzing the associated characteristic equation, we prove that the model exhibits Hopf bifurcation when time delay passes through a critical value. In order to control the undesirable Hopf bifurcation, a hybrid control strategy is proposed. By adjusting the control parameters of the strategy, it shows that the onset of Hopf bifurcation has been postponed without changing the original equilibrium point of the system. Finally, a numerical simulation is presented to verify the theoretical results.

Introduction

Bifurcation control refers to the issue of modifying the bifurcation characteristics so that achieving some desirable dynamical behaviors [1-2]. Various methods have been used to control bifurcations in both discrete and continuous systems [4-5]. For Hopf bifurcation control, much work has been used to avoid this kind of behaviors by adding different kind of controller. In[6], a static state feedback controller was proposed by Abed and Fu. In [2,3], the authors proposed a dynamic delayed feedback control method which is utilized for stabilizing unstable fixed points near Hopf bifurcation. In [7], parameters delay feedback control was used in controlling the Hopf bifurcation. Later, in order to control bifurcation and chaos in discrete nonlinear dynamical systems, a hybrid control strategy using both state feedback and parameter perturbation was put forward to control the Hopf bifurcation in [8].

Starting with the work of Watts and Strogatz [9] small-sorld networks, a lot of interesting researches on the theory and application of small-world networks have arisen [9-11]. However, the negligence of nonlinear elements such as time delay etc, the traditional model can not reflect the realistic network transmission situation. Thus, in [10], a more general nonlinear delayed differential equation model was formulated for small-world networks by Yang. Li [11] testified the local stability of one-order small-world system, and showed that Hopf bifurcation may occur as the measure parameter passes through a criticalpoint. Bifurcation periodic solutions was calculated by applying the center manifold theorem also.

Besides, in order to develop the stability of system further, Xiao et al. [12] and Zhao [13] designed a delayed feedback controller to introduce a new Hopf bifurcation with regard to measure parameter and time-delay parameter respectively as well as make the Hopf bifurcation to have certain characteristics. However, it is not obvious to obtain the change effect of stability region, there is also important to optimize the control technique of Hopf bifurcation aiming at the improvement of the control performance.

During the past decades, the study of fractional differential calculus [14] has attracted increasing interest. The applications of fractional order system have aroused exceedingly and increasingly attention not only by scientists but also by engineers in many fields, such as physics, chemical and various engineering and so on [15]. Compared with the classical integer-order models, fractional-order models provide an meaningful instrument for the description of memory and hereditary properties of various materials and processes [16]. Shi et al. [17] studied stability and

Hopf bifurcation control of a fractional-order small world network model and it has shown that small nonlinear can also result in instability of system and the onset of Hopf bifurcation.

In this paper, we attempt to apply a hybrid control strategy to a fractional order small-world network model, study the Hopf bifurcation and its stability aiming at delaying the onset of Hopf bifurcation.Thanks to the advantage of hybrid strategy that combines state feedback with parameter perturbation to realize bifurcation control without changing the equilibrium point, the performance of the original system can be retained completely. Here we choose time-delay as a bifurcation parameter.With emphasis on the relationship between the Hopf bifurcation and the time-delay, we will investigate the effect of time-delay on bifurcating behaviors in the small-world network model.

The remainder of this paper is organized as follows. In Section 2, some properties of the uncontrolled fractional order small-world network system are summarized. In Section 3, hybrid control strategy is applied to original model, and the existence of the Hopf bifurcation of this system is studied. Numerical simulations and conclusion are given to verify the theoretic analysis in Section 4 and Section 5, respectively.

Hopf bifurcation of uncontrolled System

In this section, we consider a time-delayed Hopf bifurcation in a fractional order small-world network model. The uncontrolled system can be modeled by the following delay differential equation.

$$D^d V(t) = \xi^d + V(t-\tau) - \mu \xi^d V^2(t-\tau) \tag{1}$$

where V is the total influenced volume, μ is a measure of nonlinear interactions in the network, d is the dimension of the network and the order of differential equation is $0 < d < 1$, $D^d V(t)$ is fractional-order derivative and ξ is the Newman-Watts length scale.

Let V^* be the nonzero equilibrium point of system (1).It then satisfies the following equation:

$$V^* = \frac{1+\sqrt{1+4\mu\xi^{2d}}}{2\mu\xi^d} \tag{2}$$

For convenience,the results of stability and Hopf bifurcation of system (1) are summarized here for comparison and completeness.The detailed analysis for the system can be obtained in [20].

Theorem 1 For the system (1),combing Eq.(2) we can get the following results in:
(1)When $\tau < \tau_0$, the equilibrium point of the system (1) is locally asymptotically stable;
(2)When $\tau = \tau_0$,a Hopf bifurcation occurs;
(3)When $\tau > \tau_0$,the equilibrium point of the system (1) is asymptotically unstable and a limit cycle exists.

where the critical value $\tau_0 = \dfrac{\pi - \frac{\pi d}{2}}{2d\sqrt{1+4\mu\xi^{2d}}}$.

Hybrid control of bifurcation

Now we turn to design a controller to accomplish the control of the Hopf bifurcation arising from the fractional order small-world network system (1).Equation (1) is donated as

$$D^d V(t) = g(V(t), \mu, \tau) \tag{3}$$

By described the Hybrid control strategy and adding it to the model (3),the controlled system is as follows:

$$\begin{aligned} D^d V(t) &= \alpha g(V(t), \mu, \tau) + (1-\alpha)\left(V(t-\tau) - V^*\right) \\ &= \alpha\left[\xi^d + V(t-\tau) - \mu\xi^d V^2(t-\tau)\right] + (1-\alpha)\left(V(t-\tau) - V^*\right) \end{aligned} \tag{4}$$

where α is a control parameter and $0 < \alpha < 1$.The controlled system (4) reduces to the original system

(3) if $\alpha = 1$.By selecting the appropriate control parameter α ,the Hopf bifurcation can be delayed or even eliminated completely without changing the equilibrium point of the system.So,we set $u(t) = V(t) - V^*$,the right-hand side of Eq.(4) is expanded by a Taylor expansion around the equilibrium point V^* ,we have

$$D^d V(t) = \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right) v(t - \tau) + \alpha\mu\xi^d V^2(t - \tau) \tag{5}$$

The linearized prat of system (5) is

$$D^d u(t) = (1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}) v(t - \tau) \tag{6}$$

With the initial condition at zero,applying fractional order Laplace transformation

$$\left[\zeta\left(D_t^d u(t)\right)\right](s) = s^\alpha \left[\zeta(u(t))\right](s)$$

we can get characteristic equation of Eq.(6) is

$$\lambda^d - \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right) e^{-\lambda\tau} = 0\ . \tag{7}$$

If the characteristic Eq.(7) has pure imaginary roots $\lambda = \pm i\omega, \omega > 0$,then we have

$$\begin{cases} \omega^d \cos\left(\dfrac{\pi d}{2}\right) + \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right)\cos(\omega\tau) = 0 \\ \omega^d \sin\left(\dfrac{\pi d}{2}\right) - \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right)\sin(\omega\tau) = 0 \end{cases} \tag{8}$$

From Eq.(8),we obtain

$$\omega = \sqrt[d]{1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}}\ , \tau_n = \frac{2n\pi + \pi - (\pi d / 2)}{\sqrt[d]{1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}}}, n = 0,1,2,\cdots. \tag{9}$$

In order to ensure that Eq.(7) has roots with positive real parts except for $\tau = \tau_0$,we let $\lambda = r(\cos\theta + i\sin\theta) = \sigma + i\omega, \omega > 0$,then

$$\begin{cases} r^d \cos(d\theta) + \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right) e^{-\sigma\tau} \cos(\omega\tau) = 0 \\ r^d \sin(d\theta) - \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right) e^{-\sigma\tau} \sin(\omega\tau) = 0 \end{cases} \tag{10}$$

From Eq.(11),we get $\tan(\omega\tau) = -\tan(\alpha\theta)$,and it exists a non-negative integer m to make $\omega\tau = 2m\pi + \pi - \alpha\theta$ .If $\sigma > 0$,then θ is satisfied $-\pi/2 < \theta < \pi/2$,we get

$$2m\pi + \pi - (\pi/2)d < \omega\tau < 2m\pi + \pi + (\pi/2)d \tag{11}$$

Inserting $\tau = \tau_0$,we obtain $\omega\tau < 2n\pi + \pi - (\pi/2)d$,Therefore,we can obtain a conclusion as follows:

When $n \geq 1$,it has ω in certain to make two equations of Eq.(11) come into existence at the same time.Eq.(7) has roots with positive real parts,and V^* is unstable.

When $n = 0$,there is no roots with positive real parts.That is to say,when $\tau > \tau_0$,Eq.(7) at least has one root with positive real parts,and

$$\tau_0 = \frac{\pi - \dfrac{\pi d}{2}}{\sqrt[d]{1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}}} \tag{12}$$

In order to find a Hopf bifurcation point,the following transversality condition is needed

$$\Re\left(\frac{d\lambda}{d\tau}\right)\Big|\tau = \tau_0 > 0 \tag{13}$$

Hence,let $\lambda = \pm i\omega$ be the root of Eq.(7),by calculating,we have

$$\Re\left(\frac{d\lambda}{d\tau}\right)\Big|\tau = \tau_0, \lambda = \pm i\omega = \frac{d\omega^d\sqrt{1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}}}{d^2\omega^{2d-2} + \left(1 - \alpha + \alpha\sqrt{1 + 4\mu\xi^{2d}}\right)\tau^2} > 0 \tag{14}$$

Therefore,the final condition for the occurrence of a Hopf bifurcation in the nonlinear model (4) is indeed satisfied.Thus we have the following theorem.

Theorem 2 For the controlled system (4),we can easily obtain

(1)When $\tau \in (0, \tau_0)$, the equilibrium point V^* of the controlled system (4) is locally asymptotically stable;

(2)When $\tau = \tau_0$,the controlled system (4) exists a Hopf bifurcation at equilibrium point V^*.

(3)When $\tau \in (\tau_0, \infty)$,the equilibrium point V^* of the controlled system (4) is asymptotically unstable.

Numerical simulations

In this section,we present numerical results to verify the analytic predictions obtained in the previous section,using the hybrid control strategy to control the Hopf bifurcation in a fractional order small-world network model (4).For a consistent comparison,we choose the system parameters as used in [12],with $\xi = 3$, $\mu = 0.03$.

We consider the effect of hybrid control parameter α on the stability of Eq.(4).We choose $\alpha = 1$,as we know,the system is the uncontrolled model.For the uncontrolled model,we know that when $d = 0.8$,we get $\tau_0 = 1.355$, $V^* = 15.93$.The dynamical behavior of the uncontrolled model (1) is illustrated in Fig.1 and Fig.2.It is shown that when $\tau < \tau_0$,trajectories converge to the equilibrium point,while as τ is increased to pass τ_0, V^* loses stability and a Hopf bifurcation occurs.Then,we choose $\alpha = 0.85$ the system becomes the controlled model,the critical value τ_0 increases from $\tau_0 = 1.355$ to $\tau_0 = 1.416$ .The dynamical behavior of the controlled model (4) is illustrated in Fig.3 and Fig.4.We can see that the onset of Hopf bifurcation is delayed, and the stable range in parameter space is extended.

Thus,by the hybrid control strategy,we can increase the critical value of time delay,extend the stable region of equilibrium point of controlled system and delay the onset of Hopf bifurcation.It is shown that this method can also be used in fractional order model,and easy to be applied in reality.

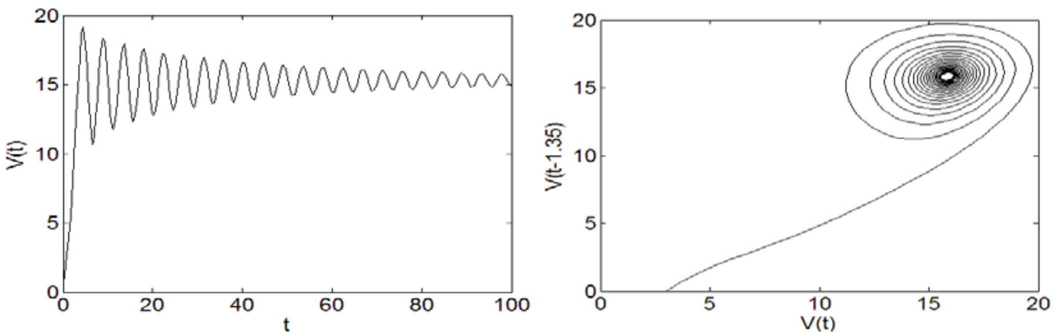

Fig. 1 Waveform graph and phase portrait of uncontrolled system with $\tau = 1.35$

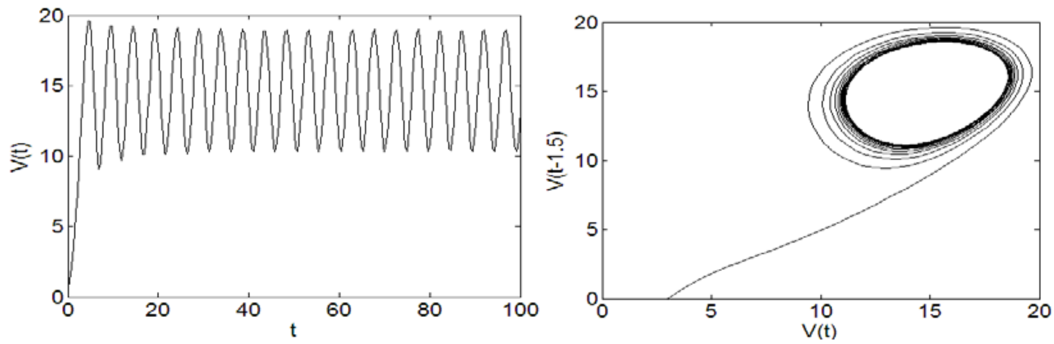

Fig. 2 Waveform graph and phase portrait of uncontrolled system with $\tau = 1.5$

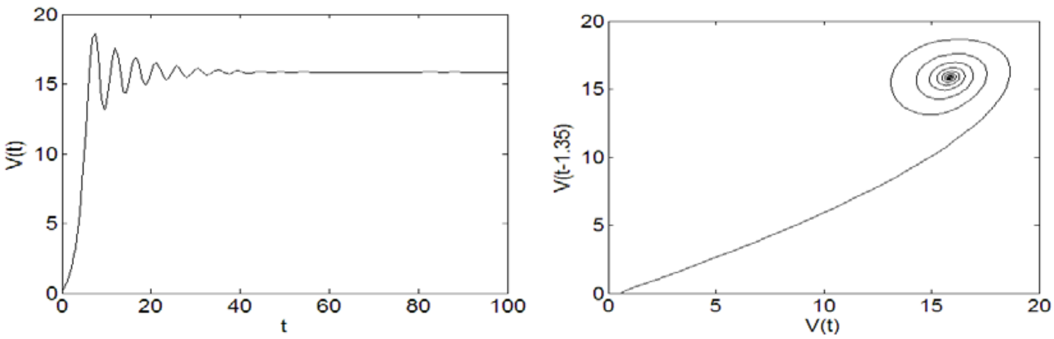

Fig. 3 Waveform graph and phase portrait of controlled system with $\tau = 1.35, \alpha = 0.85$

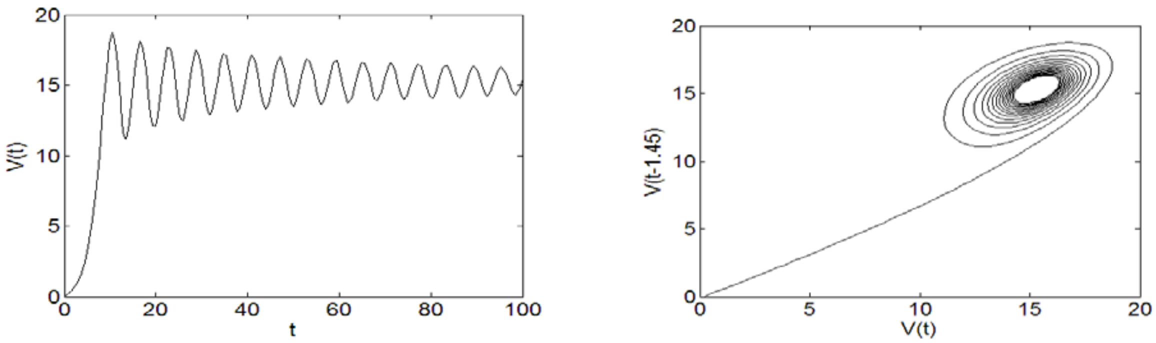

Fig. 4 Waveform graph and phase portrait of controlled system with $\tau = 1.45, \alpha = 0.85$

Conclusion

In this paper,the problem of Hopf bifurcation control for a fractional order small-world network model has been studied.To control the Hopf bifurcation,a hybrid control strategy has been proposed.By selecting appropriate control parameters,this method can effectively delay the onset of Hopf bifurcation and extend the stable region of equilibrium point of controlled system.Numurical results have validated the correctiness of the theoretical analysis.

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