A New High-Order Alternating Group Iterative Scheme

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Abstract —**In this paper, by constructing a new high-order parallel alternating group iterative algorithm, we solve the two point boundary value problem. The main method is to dividing the system of different equations into a set of subsystem which can be solved individually in parallel and sixth order accurate. We have obtained the main results and given some examples to validate our results in addition. It shows that the analysis is correct and the algorithm is feasible and efficient.**

Keywords-two point boundary value problem; sixth order; alternating group iterative scheme; convergence

I. INTRODUCTION

The solution for the two-point boundary value problems is one of the most classic problems in the numerical approximation of differential equations. By developing the alternating method, in [1-3], D. J. Evans and A. R. B. Abdullah had proposed the Alternating Group Explicit (AGE) strategy to solve the differential equations. D. J. Evans [4] also solved large linear systems by using iterative methods. This scheme is capable of parallel computation; however, its truncation error is only second order or lower. In [5], a fourth order Alternating Group Iterative (AGI) algorithm was constructed for two point boundary value problem. Thus we propose to derive a new alternating group iterative method, which has truncation error of sixth order.

II. DIFFERENCE SCHEME

We consider the periodic solution for the following second order self-adjoint ordinary differential equation (ODE):

$$Lu = -\frac{d^2u}{dx^2} + qu = f, \qquad a < x < b, \tag{1}$$

$$u(a) = u(a+H) \tag{2}$$

Here q, f are given real continuous functions on $a \le x \le b$ and $H = b - a$ represents the length of one period, with $q \ge 0$.

We partition the interval $[a,b]$ into subintervals of equal length $h = H/J$ and let the grid points be $x_i = a + (i-1)h,\ i = 1,\cdots,J+1$. u_i, q_i, f_i denote the finite difference approximations of $u(x_i), q(x_i), f(x_i)$ respectively. The parallel finite difference scheme is constructed as follows:

$$L_h u_i = \frac{-2u_{i+3} + 27u_{i+2} - 270u_{i+1} + 490u_i - 270u_{i-1} + 27u_{i-2} - 2u_{i-3}}{180h^2} + q_i u_i = f_i. \tag{3}$$

or

$$-\frac{2r}{180}u_{i+3} + \frac{27r}{180}u_{i+2} - \frac{270r}{180}u_{i+1} + (\frac{490r}{180} + q_i)u_i - \frac{270r}{180}u_{i-1} + \frac{27r}{180}u_{i-2} - \frac{2r}{180}u_{i-3} = f_i \tag{4}$$

Where $r = \frac{1}{h^2}$, $i = 1,2,\cdots,J$ .The truncation error at point $x = x_i$ for (3) is:

$$R_i = -\frac{h^6}{560}\left[\frac{d^8u}{dx^8}\right]_i + o(h^7). \tag{5}$$

We can rewrite the linear difference (3) in matrix form

$$Au = F \tag{6}$$

where

$$A = \frac{r}{360}\begin{bmatrix} 960 & -540 & 54 & -4 & & \cdots & & -4 & 54 & -540 \\ -540 & 960 & -540 & 54 & -4 & & & & -4 & 54 \\ 54 & -540 & 960 & -540 & 54 & -4 & & & & -4 \\ -4 & 54 & -540 & 960 & -540 & 54 & -4 & & & \\ & -4 & 54 & -540 & 960 & -540 & 54 & -4 & & \\ & & \ddots & & & \ddots & & & \ddots & \\ & & & -4 & 54 & -540 & 960 & -540 & 54 & -4 \\ -4 & & & & -4 & 54 & -540 & 960 & -540 & 54 \\ 54 & -4 & & & & -4 & 54 & -540 & 960 & -540 \\ -540 & 54 & -4 & & \cdots & & -4 & 54 & -540 & 960 \end{bmatrix}_{J\times J} + \begin{pmatrix} q_1 & & & & & & & \\ & q_2 & & & & & & \\ & & q_3 & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & q_{J-2} & & \\ & & & & & & q_{J-1} & \\ & & & & & & & q_J \end{pmatrix}.$$

$$u = (u_1, u_2, \cdots, u_J)^T, F = (f_1, f_2, \cdots, f_{J-2}, f_{J-1}, f_J)^T.$$

We assume that $J = 6k$, and splitting the matrix A into the sum of three matrices

$$A = G_1 + G_2 + \Sigma, \qquad (7)$$

where

$$G_1 = \frac{r}{360}\begin{bmatrix} Q_3 & & & & \\ & P_9^{(1)} & & & \\ & & \ddots & & \\ & & & Q_3 & \\ & & & & P_9^{(k)} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} q_1 & & & & \\ & q_2 & & & \\ & & \ddots & & \\ & & & q_{J-1} & \\ & & & & q_J \end{bmatrix}.$$

$$G_2 = \frac{r}{360}\begin{bmatrix} P_6 & & & & & & -4 & 54 & 540 \\ & Q_3 & & & & \cdots & & -4 & 54 \\ & & P_9^{(1)} & & & & & & -4 \\ & & & Q_3 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ -4 & & & & & & P_9^{(k-1)} & & \\ 54 & -4 & & & & & & Q_3 & \\ -540 & 54 & -4 & & \cdots & & & & P_3 \end{bmatrix},$$

in which

$$P_9^{(i)} = \begin{pmatrix} 247 & -270 & 27 & -4 & & & & & \\ -270 & 490 & -270 & 54 & -4 & & & & \\ 27 & -270 & 733 & -540 & 54 & -4 & & & \\ -4 & 54 & -540 & 980 & -540 & 54 & -4 & & \\ & -4 & 54 & -540 & 980 & -540 & 54 & -4 & \\ & & -4 & 54 & -540 & 980 & -540 & 54 & -4 \\ & & & -4 & 54 & -540 & 733 & -270 & 27 \\ & & & & -4 & 54 & -270 & 490 & -270 \\ & & & & & -4 & 54 & -270 & 247 \end{pmatrix},$$

$$i = 1, 2, \cdots, k.$$

$$P_6 = \begin{pmatrix} 980 & -540 & 54 & -4 & & \\ -540 & 980 & -540 & 54 & -4 & \\ 54 & -540 & 980 & -540 & 54 & -4 \\ -4 & 54 & -540 & 733 & -270 & 27 \\ & -4 & 54 & -270 & 490 & -270 \\ & & -4 & 54 & -270 & 247 \end{pmatrix} \quad P_3 = \begin{pmatrix} 247 & -270 & 27 \\ -270 & 490 & -270 \\ 27 & -270 & 733 \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By using (6) and (7), we have

$$(G_1 + G_2 + \Sigma)u = F \qquad (8)$$

and let us consider two equivalent forms,

$$\begin{cases} (\rho I + \overline{G}_1)\tilde{u}^{(k+1)} = \left[(\rho I - \overline{G}_2)u^{(k)} + F\right] \\ (\rho I + \overline{G}_2)u^{(k+1)} = \left[(\rho I - \overline{G}_1)\tilde{u}^{(k+1)} + F\right] \end{cases} k = 0, 1, 2, \cdots, \qquad (9)$$

where $\overline{G}_1 = G_1 + \frac{1}{2}\Sigma, \overline{G}_2 = G_2 + \frac{1}{2}\Sigma,$ ρ is a positive iteration parameter and

$(\rho I + \overline{G}_1)^{-1}$, $(\rho I + \overline{G}_2)^{-1}$ exist.

III. THE CONVERGENCE ANALYSIS

Lemma 1 If $A = (a_{ij}) \in C^{m \times n}$, then $\|A\|_2 = \sqrt{\lambda_1}$, λ_1 is the max eigenvalue of $A^H A$.

According to lemma 1, if A is real and symmetric matrix, then $\|A\|_2 = \rho(A)$, $\rho(A)$ is the spectral radius of A .

Lemma 2 In (8), $\overline{G}_1$ and $\overline{G}_2$ are both positive definite matrices. Eliminating $\tilde{u}^{(k+1)}$ of (9), we have

$$u^{(k+1)} = Tu^{(k)} + \tilde{F}. \qquad (10)$$

here

$$T = (\rho I + \overline{G}_2)^{-1}(\rho I - \overline{G}_1)(\rho I + \overline{G}_1)^{-1}(\rho I - \overline{G}_2)$$

$$\tilde{F} = (\rho I + \overline{G}_2)^{-1}\left[(\rho I - \overline{G}_1)(\rho I + \overline{G}_1)^{-1}F + F\right].$$

Now define the matrix:

$$\tilde{T} = (\rho I + \overline{G}_2)T(\rho I + \overline{G}_2)^{-1},$$

which is similar to T and thus has the same eigenvalues as T .Then it is evident that:

$$\rho(T) = \rho(\tilde{T}) = \|\tilde{T}\|_2 \le$$
$$\|(\rho I - \overline{G}_1)(\rho I + \overline{G}_1)^{-1}\|_2 \times \|(\rho I - \overline{G}_2)(\rho I + \overline{G}_2)^{-1}\|_2$$
$$= \max_i \left|\frac{\rho - \mu_i}{\rho + \mu_i}\right| \cdot \max_i \left|\frac{\rho - \eta_i}{\rho + \eta_i}\right|.$$

Here μ_i, η_i $(i = 1, 2, \cdots, J)$ are respectively the eigenvalues of $\overline{G}_1$ and $\overline{G}_2$ .According to the lemmas, $\overline{G}_1$ and $\overline{G}_2$ are positive definite matrices, therefore $\mu_i > 0, \eta_i > 0$ $(i = 1, 2, \cdots, J)$ and we have $\rho(T) < 1$. Hence the convergence of the alternating iterative scheme is proved.

IV. NUMERICAL EXPERIMENT

We perform the numerical simulations using the following model problem:

$$\begin{cases} -\dfrac{d^2u}{dx^2} + xu = \sin x(x+1) \quad 0 < x < 2\pi, \\ u(0) = u(2\pi). \end{cases} \qquad (11)$$

The exact solution is:

$$u = \sin x \qquad (12)$$

The error bounds of the sixth order scheme are examined first. The L^2 errors are defined to be $e_h = \|u - u(x)\|_{L^2}$, and are computed based on the following four different ρ, and four different h:

$$\rho = 1,3,5,7\,; \quad h = \frac{2\pi}{12}, \frac{2\pi}{30}, \frac{2\pi}{60}, \frac{2\pi}{120}.$$

The results are listed in Table 1 .It is not hard to see that the L^2 errors of the six-order scheme are order 6 in space, which comfirms the earlier results in Section 2.

Next, we compare the accuracy of the sixth order AGI scheme to the fourth order AGI scheme. The computation is based on the same ρ, and h is taken to be $\pi/12$ .The absolute errors(ae) and the percentage errors(pe) for these two schemes are listed in Table 2 and plotted in Figure 1 and Figure 2 at the end of paper. Evidently, the results show that the sixth order AGI scheme is more accurate than the fourth order AGI scheme .

TABLE I. CONVERGENCE RATES FOR THE SIXTH ORDER SCHEME

J	$\rho=1$		$\rho=3$		$\rho=5$		$\rho=7$	
	e_h*10^5	e_h/h^6	e_h*10^7	e_h/h^6	e_h*10^7	e_h/h^6	e_h*10^7	e_h/h^6
12	3.012	0.001	301	0.001	301	0.001	301	0.001
30	0.020	0.002	1.98	0.002	1.98	0.002	1.98	0.002
60	1.579	11.98	0.044	0.003	0.044	0.003	0.44	0.003
120	146	1463	0.017	8.21	0.001	0.005	0.0009 6	0.005

TABLE II. ABSOLUTE AND PERCENTAGE ERRORS

	$J=24, \Delta x = \pi/12, \rho = 2$					
	$x=0.26$	$x=1.05$	$x=2.10$	$x=3.40$	$x=4.20$	$x=6.02$
6th-order AGI ae	$.133\times10^{-6}$	$.244\times10^{-6}$	$.187\times10^{-6}$	$.174\times10^{-7}$	$.890\times10^{-7}$	$.190\times10^{-7}$
pe	$.517\times10^{-6}$	$.282\times10^{-6}$	$.216\times10^{-6}$	$.673\times10^{-7}$	$.103\times10^{-6}$	$.732\times10^{-7}$
4th-order AGI ae	$.122\times10^{-4}$	$.222\times10^{-4}$	$.170\times10^{-4}$	$.159\times10^{-5}$	$.811\times10^{-5}$	$.172\times10^{-5}$
pe	$.471\times10^{-4}$	$.256\times10^{-4}$	$.196\times10^{-4}$	$.614\times10^{-4}$	$.936\times10^{-5}$	$.664\times10^{-5}$
Exact solution	.259	.866	.866	-.259	-.866	-.259

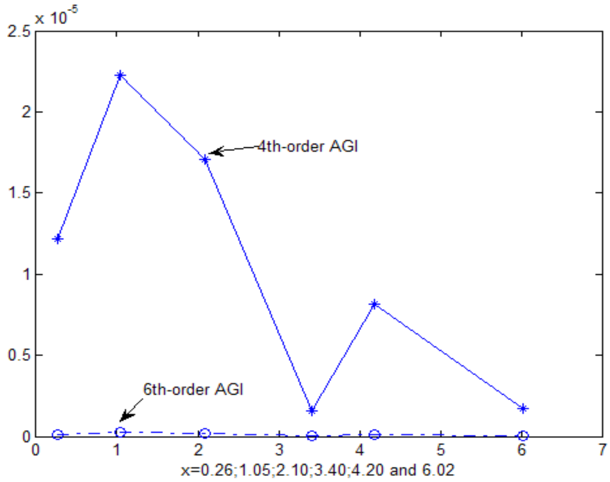

FIGURE I. ABSOLUET ERRORS $J = 24, \rho = 2$

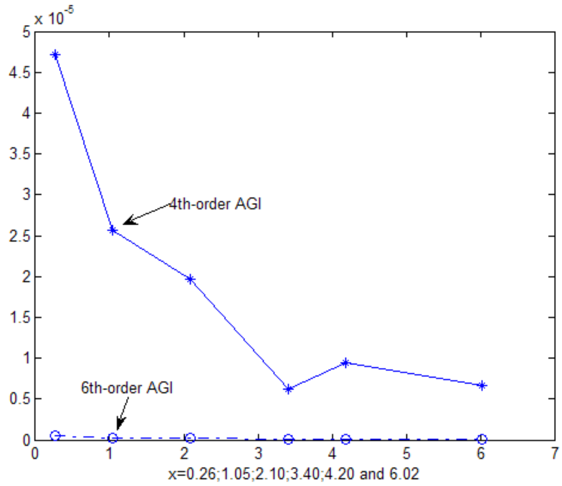

FIGURE II. PERCENTAGE ERRORS $J = 24, \rho = 2$

V. CONCLUSIONS

In this paper, a sixth order alternating group iterative algorithm is derived for two point boundary value problem. The scheme has truncation error of sixth order in space which is higher than similar fourth order AGI scheme. The scheme is proved to be stable under reasonable condition. Numerical example is also prensented.

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