Oscillation of Nonlinear Impulsive Delay Hyperbolic Equation with Functional Arguments via Riccati Method

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Abstract —**In this paper we mainly deal with the oscillation problem of nonlinear impulsive hyperbolic equation with functional arguments by using integral averaging method and a generalized Riccati technique. A sufficient condition for oscillation of the solutions of nonlinear impulsive hyperbolic equation with functional arguments is obtained.**

Keywords-hyperbolic equation; oscillation; Riccati method; impulsive; delay

I. Introduction

The theories of nonlinear partial functional differential equations are applied in many fields. In recent years the research of oscillation to impulsive partial differential problems has caught more and more attention. In this paper, we study the oscillation property of the impulsive delay hyperbolic equation

$$\frac{\partial}{\partial t}(r(t)\frac{\partial}{\partial t}u(x,t)) = a(t)h(u(x,t))\Delta u(x,t) - \sum_{i=1}^{n} b_i(t)h_i(u(x,\tau_i(t)))\Delta u(x,\tau_i(t))$$
$$+\sum_{j=1}^{m} q_j(x,t)\varphi_j(u(x,t)),\quad t\neq t_k,\quad (x,t)\in\Omega\equiv G\times(0,+\infty), \tag{1}$$
$$u(x,t_k^+) - u(x,t_k^-) = \alpha_k u(x,t_k)\quad t=t_k, k=1,2,\cdots, \tag{2}$$
$$u_t(x,t_k^+) - u_t(x,t_k^-) = \beta_k u_t(x,t_k)\quad t=t_k, k=1,2,\cdots, \tag{3}$$

where G is a bounded domain of $\mathbb{R}^n$ with the smooth boundary ∂G. We consider the following boundary condition:

$$u=0\quad (x,t)\in\partial G\times[0,+\infty). \tag{4}$$

Following are the basic hypotheses:

(H1) $r(t)\in C([0,+\infty);(0,+\infty))$, $a(t), b_i(t)\in PC([0,+\infty);[0,+\infty)), i=1,2,\cdots n,$

$q_j(x,t)\in C(\bar{\Omega};[0,+\infty)),\quad j=1,2,\cdots,m,$ where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind only at $t=t_k, k=1,2,\cdots$.

(H2) $\tau_i(t)\in C([0,+\infty);R), \lim_{t\to+\infty}\tau_i(t)=+\infty, i=1,2,\cdots,n$.

(H3) $h'(u), h_i'(u)\in C(R,R)$, $\varphi_j(s)\in C(R,R)$, $\alpha_k, \beta_k = const. > -1$, $uh'(u)\geq 0$, $uh_i'(u)\geq 0$, $h(0)=0$, $h_i(0)=0$, $i=1,2,\cdots,n$, $\frac{\varphi_j(s)}{s}\geq C_j = const. > 0$ for $s\neq 0$. $0<t_1<t_2<\cdots<t_k<\cdots$, $\lim_{t\to+\infty} t_k = +\infty$, $k=1,2,\cdots$.

We introduce the notations: $U(t)=\int_G u(x,t)dx$ and $q_j(t)=\min_{x\in\bar{G}} q_j(x,t)$.

Definition 1.1. *By a solution $u(x,t)$ of problem (1)-(4) we mean a function $u(x,t)\in C^2(\bar{G}\times[t_{-1},\infty))$ which satisfies problem (1)-(4), where*

$$t_{-1}=\min\left\{0,\min_{1\leq i\leq n}\left\{\inf_{t\geq 0}\tau_i(t)\right\}\right\}.$$

Definition 1.2. *The solution $u(x,t)$ of problem (1)-(4) is said to be non- oscillatory in domain Ω if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.*

Definition 1.3. *We say that functions (H_1,H_2) belong to a function class $\mathcal{H}$, denoted by $(H_1,H_2)\in\mathcal{H}$, if $(H_1,H_2)\in C(D;[0,+\infty))$ satisfy*

$$H_i(t,t)=0,\ H_i(t,s)>0\ (i=1,2)\ \text{for}\ t>s,$$

where $D=\{(t,s):0<s\leq t<+\infty\}$. Moreover, the partial derivatives $\partial H_1/\partial t$ and $\partial H_2/\partial s$ exist on D such that

$$\frac{\partial H_1}{\partial t}(s,t)=h_1(s,t)H_1(s,t)\ \text{and}\ \frac{\partial H_2}{\partial s}(t,s)=-h_2(t,s)H_2(t,s),$$

where $h_1,h_2\in C_{loc}(D;\mathbb{R})$.

In recent years, there has been much research activity concerning the oscillation theory of nonlinear hyperbolic equations with functional arguments by employing Riccati technique. Riccati techniques were used to obtain various oscillation results. Recently, Y.Shoukaku and N. Yoshida [2] derived oscillation criteria by using oscillation criteria of Riccati inequality. In this work, we study the hyperbolic equation with impulsive.

II. MAIN RESULTS

Theorem 2.1. *If for each* $T \ge 0$, *there exist* $(H_1, H_2) \in \mathcal{H}$ *and* $a, b, c \in \mathbb{R}$ *such that* $T \le a < c < b$ *and*

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\prod_{t_1 \le t_k < s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\left(C_l q_l(s) - \frac{1}{4}r(s)\lambda_1^2(s,a)\right)\psi(s)ds$$
$$+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\prod_{t_1 \le t_k < s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\left(C_l q_l(s) - \frac{1}{4}r(s)\lambda_2^2(b,s)\right)\psi(s)ds > 0, \tag{5}$$

then (1)-(4) has no eventually positive solution, where $\psi(t) \in C^1((T_0, +\infty); (0, +\infty))$ *for some* $T_0 > 0$ *and*

$$\lambda_1(s,t) = \frac{\psi'(s)}{\psi(s)} + h_1(s,t),$$
$$\lambda_2(t,s) = \frac{\psi'(s)}{\psi(s)} - h_2(t,s).$$

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the problem (1) - (4). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, +\infty)$ for some $t_0 > 0$ because the case where $u(x,t) < 0$ can be treated similarly. Since (H2) holds, we see that $u(x, \tau_i(t)) > 0$ $(i = 1, 2, \cdots n)$ in $G \times [t_1, +\infty)$ for some $t_1 \ge t_0$.

(1) For $t \ge t_1, t \ne t_k, k = 1, 2, \cdots$, integrating (1) with respect to x over G, we obtain

$$\frac{d}{dt}\left(r(t)\int_G \frac{\partial}{\partial t}u(x,t)dx\right) = a(t)\int_G h(u(x,t))\Delta u(x,t)dx - \sum_{j=1}^m \int_G q_j(x,t)\varphi_j(u(x,t))dx$$
$$+\sum_{i=1}^n b_i(t)\int_G h_i(u(x,\tau_i(t)))\Delta u(x,\tau_i(t))dx.$$

By Green's formula and the boundary condition, we have

$$\int_G h(u(x,t))\Delta u(x,t)dx = \int_{\partial G} h(u(x,t))\frac{\partial u(x,t)}{\partial n}ds - \int_G h'(u)\,|\,grad u\,|^2\, dx$$
$$= -\int_G h'(u)\,|\,grad u\,|^2\, dx \le 0.$$
$$\int_G h_i(u(x,\tau_i(t)))\Delta u(x,\tau_i(t))dx \le 0.$$

For condition (H3) we can easily obtain

$$\int_G q_j(x,t)\varphi_j(u(x,t))dx \ge C_j q_j(t)\int_G u(x,t)dx.$$

Then $U(t) > 0$, and it follows that

$$(r(t)U'(t))' + \sum_{j=1}^m C_j(t)q_j(t)U(t) \le 0.$$

For some $l \in \{1, 2, \cdots, m\}$, we can get

$$(r(t)U'(t))' + C_l q_l(t)U(t) \le 0, \ \ t \ge t_1, t \ne t_k.$$

(2) For $t = t_k$, $k = 1, 2, \cdots$. From (2)-(3) we have that

$$\int_G u(x,t_k^+)dx - \int_G u(x,t_k^-)dx = \alpha_k\int_G u(x,t_k),$$

$$\int_G u_t(x,t_k^+)dx - \int_G u_t(x,t_k^-)dx = \beta_k\int_G u_t(x,t_k),$$

that is

$$U(t_k^+) = (1+\alpha_k)U(t_k), U'(t_k^+) = (1+\beta_k)U'(t_k).$$

Thus we obtain that the functions $U(t)$ is a eventually positive solution of the impulsive differential inequality

$$\begin{cases} (r(t)y'(t))' + C_l q_l(t)y(t) \le 0, \\ y(t_k^+) = (1+\alpha_k)y(t_k), \\ y'(t_k^+) = (1+\beta_k)y'(t_k). \end{cases} \tag{6}$$

Set $w(t) = \frac{r(t)U'(t)}{U(t)}$ for $t \ge t_1$. From (6), we obtain that

$$w'(t) + \frac{1}{r(t)}w^2(t) \le -C_l q_l(t),$$

$$w(t_k^+) = \frac{1+\beta_k}{1+\alpha_k}w(t_k)\cdot$$

Define $v(t) = \prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}w(t)$. In fact, $w(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in view of $w(t_k^+) = \frac{1+\beta_k}{1+\alpha_k}w(t_k)$, it follows that for $t \ge t_1$

$$v(t_k^+) = \prod_{t_1 \le t_j \le t_k}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}w(t_k^+) = \prod_{t_1 \le t_j < t_k}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}w(t_k) = v(t_k),$$

and for all $t \ge t_1$

$$v(t_k^-) = \prod_{t_1 \le t_j \le t_{k-1}}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}w(t_k^-) = \prod_{t_1 \le t_j < t_k}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}w(t_k) = v(t_k),$$

which implies that $v(t)$ is continuous on $[t_1, +\infty)$.

$$v'(t) + \prod_{t_1 \le t_k < t}\frac{1+\beta_k}{1+\alpha_k}\frac{1}{r(t)}v^2(t) + \prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}C_l q_l(t)$$
$$= \prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}w'(t) + \frac{1}{r(t)}\prod_{t_1 \le t_k < t}\frac{1+\beta_k}{1+\alpha_k}\left(\prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\right)^2 w^2(t) + \prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}C_l q_l(t)$$
$$= \prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\left[w'(t) + \frac{1}{r(t)}w^2(t) + C_l q_l(t)\right] \le 0.$$

That is to say

$$v'(t) + \prod_{t_1 \le t_k < t}\frac{1+\beta_k}{1+\alpha_k}\frac{1}{r(t)}v^2(t) \le -\prod_{t_1 \le t_k < t}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}C_l q_l(t). \tag{7}$$

Multiplying (7) by $\psi(s)$, we obtain

$$\prod_{t_1 \le t_k < s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}\psi(s)C_l q_l(s) \le -\psi(s)v'(s) - \prod_{t_1 \le t_k < s}\frac{1+\beta_k}{1+\alpha_k}\frac{\psi(s)}{r(s)}v^2(s) \tag{8}$$

Multiplying (8) by $H_2(t,s)$ and integrating over $[c,t]$ for $t \in [c,b)$, we have

$$\int_c^t \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} H_2(t,s)\psi(s) C_l q_l(s) ds$$
$$\le -\int_c^t H_2(t,s)\psi(s)v'(s)ds - \int_c^t H_2(t,s) \prod_{t_1 \le t_k < s} \frac{1+\beta_k}{1+\alpha_k} \frac{\psi(s)}{r(s)} v^2(s) ds$$
$$= H_2(t,c)v(c)\psi(c)$$
$$-\int_c^t H_2(t,s)(\sqrt{\frac{\prod_{t_1 \le t_k < s}\frac{1+\beta_k}{1+\alpha_k}}{r(s)}} v(s) - \frac{1}{2}\lambda_2(t,s)\sqrt{\frac{r(s)}{\prod_{t_1 \le t_k < s}\frac{1+\beta_k}{1+\alpha_k}}})^2 \psi(s) ds$$
$$+ \frac{1}{4}\int_c^t \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} H_2(t,s) r(s)\psi(s)\lambda_2^2(t,s) ds$$
$$\le H_2(t,c)v(c)\psi(c) + \frac{1}{4}\int_c^t \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} H_2(t,s) r(s)\psi(s)\lambda_2^2(t,s) ds,$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)(\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} C_l q_l(s) - \frac{1}{4}\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} r(s)\lambda_2^2(t,s))\psi(s) ds$$
$$\le v(c)\psi(c).$$

Letting $t \to b^-$ in the above, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)(\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} C_l q_l(s) - \frac{1}{4}\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} r(s)\lambda_2^2(b,s))\psi(s) ds$$
$$\le v(c)\psi(c). \tag{9}$$

On the other hand, multiplying (8) by $H_1(s,t)$ and integrating over $[t,c]$ for $t \in (a,c]$, we obtain

$$\int_t^c \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} H_1(s,t)\psi(s) C_l q_l(s) ds$$
$$\le -\int_t^c H_1(s,t)\psi(s)v'(s)ds - \int_t^c H_1(s,t) \prod_{t_1 \le t_k < s} \frac{1+\beta_k}{1+\alpha_k} \frac{\psi(s)}{r(s)} v^2(s) ds$$
$$= -H_1(c,t)v(c)\psi(c)$$
$$-\int_t^c H_1(s,t)(\sqrt{\frac{\prod_{t_1 \le t_k < s}\frac{1+\beta_k}{1+\alpha_k}}{r(s)}} v(s) - \frac{1}{2}\lambda_1(s,t)\sqrt{\frac{r(s)}{\prod_{t_1 \le t_k < s}\frac{1+\beta_k}{1+\alpha_k}}})^2 \psi(s) ds$$
$$+ \frac{1}{4}\int_t^c \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} H_1(s,t) r(s)\psi(s)\lambda_1^2(s,t) ds$$
$$\le -H_1(c,t)v(c)\psi(c) + \frac{1}{4}\int_t^c \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} H_1(s,t) r(s)\psi(s)\lambda_1^2(s,t) ds,$$

and so

$$\frac{1}{H_1(c,t)}\int_t^c H_1(s,t)(\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} C_l q_l(s) - \frac{1}{4}\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} r(s)\lambda_1^2(s,t))\psi(s) ds$$
$$\le -v(c)\psi(c).$$

Letting $t \to a^+$ in the above, we get

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)(\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} C_l q_l(s) - \frac{1}{4}\prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} r(s)\lambda_1^2(s,a))\psi(s) ds$$
$$\le -v(c)\psi(c). \tag{10}$$

Adding (9) and (10), we easily obtain the following:

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a) \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} (C_l q_l(s) - \frac{1}{4} r(s)\lambda_1^2(s,a))\psi(s) ds$$
$$+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s) \prod_{t_1 \le t_k < s} (\frac{1+\beta_k}{1+\alpha_k})^{-1} (C_l q_l(s) - \frac{1}{4} r(s)\lambda_2^2(b,s))\psi(s) ds \le 0,$$

which contradicts the condition (5).

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