Uncertain Concepts in a Formal Context

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Abstract—**Formal concept analysis (FCA) provides a theoretical framework for learning hierarchies of knowledge clusters. This paper is devoted to the study of uncertainty in FCA. We introduce the notions of uncertain concept and its accuracy degree in FCA, and present approaches to characterizing accuracy degree of uncertain concepts from approximation operators and covering rough set points of views.**

Keywords-formal concept analysis; formal concept; uncertain concept; accuracy degree

I. INTRODUCTION

Formal concept analysis (FCA) was independently introduced by Wille in the 1980's [1]. FCA deals with relational information structures (formal contexts) derived from human investigation (judgment, observation, measure, etc) and provide a theoretical framework for learning hierarchies of knowledge clusters called formal concepts. As an efficient tool of data analysis and knowledge processing, FCA has been applied in many fields, such as knowledge engineering, data mining, information searches, and software engineering [1, 2].

Works on FCA are progressing rapidly. It has become increasingly popular among various methods of conceptual data analysis and knowledge representation. Most of the researches on FCA concentrate on such topics as: construction of the concept lattice [3, 4], pruning of the concept lattice [5, 6], acquisition of rules [7], relationship between the concept lattice and rough set [8, 9], and applications [10, 11]. The combination of FCA and fuzzy set theory is another important issue. Many efforts have been made to compare and combine these two theories. Several fuzzy extensions of formal concept analysis have been addressed in the literature. A central notion in existing approaches is the so called fuzzy formal context whose entries become now degrees from a totally ordered set L (generally $L=[0,1]$), whereby property satisfaction becomes a matter of degree [12-15].

In the present paper, we attempt to conduct a further study on the uncertainty in a formal context. In FCA, formal concept is a key notion. It is defined by an (set of objects, set of attributes) pair. From extent point of view, these sets of objects are (exact) concepts. In this paper, we introduce the notions of uncertain concept and its accuracy degree in FCA, and present approaches to characterizing accuracy degree. The paper is organized as follows: In Section 2, we recall some notions and properties of FCA. In Section 3, the notions of uncertain concept and its accuracy degree in FCA are proposed. Based on approximation operators, the accuracy degree of uncertain concept is presented. Furthermore, in Section 4, we construct another kind of accuracy degree from covering rough set point of view. The paper is completed with some concluding remarks.

II. OVERVIEW OF FORMAL CONCEPT ANALYSIS

Formal Concept Analysis [1] provides a theoretical framework for learning hierarchies of knowledge clusters called formal concepts. A basic notion in FCA is the formal context. Given a set G of objects and a set M of attributes (also called properties), a formal context consists of a triple $\kappa=(G,M,I)$ where I specifies (Boolean) relationships between objects of G and attributes of M, i.e., $I\subseteq G\times M$ .Usually, formal contexts are given under the form of a table that formalizes these relationships. A table entry indicates whether an object has the attribute (this is denoted by a 1), or not (it is often indicated by 0). Let $I(g)=\{m\in M;(g,m)\in I\}$ be the set of attributes satisfied by object g, and let $I(m)=\{g\in G;(g,m)\in I\}$ be the set of objects that satisfy the attribute m.

Given a formal context $\kappa=(G,M,I)$. We define the set-valued operators $\uparrow: P(G)\to P(M)$, and $\downarrow: P(M)\to P(G)$ as follows [1]: for each $A\in P(G)$ and $B\in P(M)$,

$$A^{\uparrow}=\{m\in M;\forall g\in A((g,m)\in I)\} \tag{1}$$

$$B^{\downarrow}=\{g\in G;\forall m\in B((g,m)\in I)\} \tag{2}$$

That is to say, $A^{\uparrow}$ is the set of all attributes which is satisfied all objects in A, whereas $B^{\downarrow}$ is the set of all objects which satisfies all attributes in B. A formal concept of κ is defined as a pair (A,B) with $A\subseteq G$, $B\subseteq M$, $A^{\uparrow}=B$ and $B^{\downarrow}=A$. A is called the extent of the formal concept (A,B), whereas B is called the intent.

The main problem in formal concept analysis is that of extracting formal concepts from object/attribute relations. The set of all formal concepts equipped with a partial order (denoted by $\le$) defined as: $(X_1,Y_1)\le(X_2,Y_2)$ if and only if $X_1\subseteq X_2$ (or equivalently, $Y_2\subseteq Y_1$), forms a complete lattice, called the concept lattice of κ and denoted by $L(\kappa)$. Its structure is given by the following theorem.

Theorem 1 [1] The concept lattice $L(\kappa)$ is a complete lattice in which infimum and supremum are given by:

$$\wedge_{j\in J}(X_j,Y_j)=(\bigcap_{j\in J}X_j,(\bigcup_{j\in J}Y_j)^{\downarrow\uparrow}) \tag{3}$$

$$\vee_{j\in J}(X_j,Y_j)=((\bigcup_{j\in J}X_j)^{\downarrow\uparrow},\bigcap_{j\in J}Y_j) \quad (4)$$

For a formal context $\kappa=(G,M,I)$, the following properties can be easily proved [1]: for any $X, X_1, X_2 \subseteq G$, $Y, Y_1, Y_2 \subseteq M$,

(1) $X_1 \subseteq X_2 \Rightarrow X_2^{\uparrow} \subseteq X_1^{\uparrow}$, $Y_1 \subseteq Y_2 \Rightarrow Y_2^{\downarrow} \subseteq Y_1^{\downarrow}$.

(2) $(X_1 \cup X_2)^{\uparrow} = X_1^{\uparrow} \cap X_2^{\uparrow}$, $(Y_1 \cup Y_2)^{\downarrow} = Y_1^{\downarrow} \cap Y_2^{\downarrow}$.

(3) $(X_1 \cap X_2)^{\uparrow} \supseteq X_1^{\uparrow} \cup X_2^{\uparrow}$, $(Y_1 \cap Y_2)^{\downarrow} \supseteq Y_1^{\downarrow} \cup Y_2^{\downarrow}$.

(4) $X \subseteq X^{\uparrow\downarrow}$, $Y \subseteq Y^{\downarrow\uparrow}$.

(5) $X^{\uparrow} = X^{\uparrow\downarrow\uparrow}$, $Y^{\downarrow} = Y^{\downarrow\uparrow\downarrow}$.

(6) $(X^{\uparrow\downarrow}, X^{\uparrow})$ and $(Y^{\downarrow}, Y^{\downarrow\uparrow})$ are all formal concepts.

Example 1 [16] We consider the following formal context $\kappa=(G,M,I)$, where $G=\{1,2,3,4,5,6,7,8\}$, $M=\{a,b,c,d,e,f,g,h,i\}$, and I is represented in the following table.

TABLE I. THE FORMAL CONTEXT $\kappa=(G,M,I)$.

	a	b	c	d	e	f	g	h	i
1	1	1	0	0	0	0	1	0	0
2	1	1	0	0	0	0	1	1	0
3	1	1	1	0	0	0	1	1	0
4	1	0	1	0	0	0	1	1	1
5	1	1	0	1	0	1	0	0	0
6	1	1	1	1	0	1	0	0	0
7	1	0	1	1	1	0	0	0	0
8	1	0	1	1	0	1	0	0	0

The formal concepts in this formal context are listed as follows:

$(\emptyset,\{a,b,c,d,e,f,g,h,i\})$, $(\{3\},\{a,b,c,g,h\})$, $(\{4\},\{a,c,g,h,i\})$,

$(\{6\},\{a,b,c,d,f\})$, $(\{7\},\{a,c,d,e\})$, $(\{2,3\},\{a,b,g,h\})$,

$(\{3,4\},\{a,c,g,h\})$, $(\{3,6\},\{a,b,c\})$, $(\{5,6\},\{a,b,d,f\})$, $(\{6,8\},\{a,c,d,f\})$,

$(\{1,2,3\},\{a,b,g\})$, $(\{2,3,4\},\{a,g,h\})$, $(\{5,6,8\},\{a,d,f\})$,

$(\{6,7,8\},\{a,c,d\})$, $(\{1,2,3,4\},\{a,g\})$, $(\{5,6,7,8\},\{a,g\})$,

$(\{1,2,3,5,6\},\{a,b\})$, $(\{3,4,6,7,8\},\{a,c\})$, $(\{1,2,3,4,5,6,7,8\},\{a\})$.

III. UNCERTAIN CONCEPT IN FCA

Let $\kappa=(G,M,I)$ be a formal context. For a formal concept (X,Y), the extent X and intent Y uniquely determine each other. Actually, we have $L(\kappa)=\{(X^{\uparrow\downarrow},X^{\uparrow});X\subseteq G\}$. That is to say, every formal concept can be described from extent point of view. In what follows, we denote $L(\kappa')=\{X^{\uparrow\downarrow};X\subseteq G\}$. $L(\kappa')$ can be looked upon the set of all formal concepts with respect to κ. Clearly, we have

Theorem 2 $L(\kappa')$ is a cover of set G of objects. That is to say, $\bigcup_{X\subseteq G} X^{\uparrow\downarrow}=G$.

Definition 1 Let $\kappa=(G,M,I)$ be a formal context. For every $X\subseteq G$, the lower approximation X_* and upper approximation X^* of X are defined respectively by

$$X_*=\bigcup\{Y\in L(\kappa');Y\subseteq X\} \quad (5)$$

$$X^*=\bigcap\{Y\in L(\kappa');X\subseteq Y\} \quad (6)$$

Theorem 3 Let $\kappa=(G,M,I)$ be a formal context and $X\subseteq G$. $X_*=X^*$ if and only if $X^{\uparrow\downarrow}=X$.

Proof. Assume that $X^{\uparrow\downarrow}=X$. By $X^{\uparrow\downarrow}\in L(\kappa')$, we have $X\in L(\kappa')$ and hence $X_*=\bigcup\{Y\in L(\kappa');Y\subseteq X\}=X$, $X^*=\bigcap\{Y\in L(\kappa');X\subseteq Y\}=X$ and consequently $X_*=X^*$. Conversely, we assume that $X_*=X^*$. By $X_*\subseteq X\subseteq X^*$, it follows that $X=X^*=\bigcap\{Y\in L(\kappa');X\subseteq Y\}$. Thus we have
$X^{\uparrow}=(\bigcap\{Y\in L(\kappa');X\subseteq Y\})^{\uparrow}\supseteq\bigcup\{Y^{\uparrow};Y\in L(\kappa');X\subseteq Y\}$,
$X^{\uparrow\downarrow}\subseteq(\bigcup\{Y^{\uparrow};Y\in L(\kappa');X\subseteq Y\})^{\downarrow}=\bigcap\{Y^{\uparrow\downarrow};Y\in L(\kappa');X\subseteq Y\}$
$=\bigcap\{Y;Y\in L(\kappa');X\subseteq Y\}=X$.

Consequently, we have $X^{\uparrow\downarrow}=X$.

Definition 2 Let $\kappa=(G,M,I)$ be a formal context and $X\subseteq G$. If $X=X^{\uparrow\downarrow}$, i.e., X is the extent of a formal concept, then X is called an exact concept; otherwise X is called a uncertain concept.

Definition 3 Let $\kappa=(G,M,I)$ be a formal context and $X\subseteq G$. The accuracy degree $Ad(X)$ of X is defined as $Ad(X)=\frac{|X_*|}{|X^*|}$, where $|X_*|$ and $|X^*|$ are cardinality of X_* and X^* respectively.

By Theorem 3, X is an exact concept, if and only if $X_*=X^*$. Thus we have the following corollary:

Corollary 1 Let $\kappa=(G,M,I)$ be a formal context and $X\subseteq G$.

(1) $0\le Ad(X)\le 1$.

(2) $Ad(X)=1$ if and only if X is an exact concept.

Theorem 4 Let $\kappa=(G,M,I)$ be a formal context and

$X \subseteq G$. Then $X^{*} \in L(\kappa')$.

Proof. By the definition, we have $X^{*} = \bigcap\{Y \in L(\kappa'); X \subseteq Y\}$. It follows that $X^{*\uparrow} = (\bigcap\{Y \in L(\kappa'); X \subseteq Y\})^{\uparrow} \supseteq \bigcup\{Y^{\uparrow}; Y \in L(\kappa'); X \subseteq Y\}$,

$$X^{*\uparrow\downarrow} \subseteq (\bigcup\{Y^{\uparrow}; Y \in L(\kappa'); X \subseteq Y\})^{\downarrow} = \bigcap\{Y^{\uparrow\downarrow}; Y \in L(\kappa'); X \subseteq Y\}$$
$$= \bigcap\{Y; Y \in L(\kappa'); X \subseteq Y\} = X^{*}.$$

Thus we have $X^{*\uparrow\downarrow} = X^{*}$ and hence $X^{*} \in L(\kappa')$.

This theorem shows that, for every $X \subseteq G$, there is the least object in the set $\{Y \in L(\kappa'); X \subseteq Y\}$ (with respect to set inclusion relation $\subseteq$). The following example shows that for the set $\{Y \in L(\kappa'); X \subseteq Y\}$, the greatest objects need not exist.

Example 2 We consider the formal context $\kappa = (G, M, I)$ in Example 1. Let $X = \{3,4,6\}$. It follows that

$\{Y \in L(\kappa'); X \subseteq Y\} = \{\{1,2,3,4,5,6,7,8\}\}$,

$X^{*} = \bigcap\{Y \in L(\kappa'); X \subseteq Y\} = \{1,2,3,4,5,6,7,8\}$.

Thus X^{*} is an exact concept. Similarly, we have

$\{Y \in L(\kappa'); Y \subseteq X\} = \{\{3\},\{4\},\{6\},\{3,4\},\{3,6\}\}$.

It has no greatest objects. Furthermore, $X_{*} = \bigcup\{Y \in L(\kappa'); Y \subseteq X\} = X$ and X is not an exact concept. Consequently, $Ad(X) = \frac{|X_{*}|}{|X^{*}|} = \frac{3}{8}$.

The following theorem shows that $L(\kappa')$ is a topological base of a topology on G.

Theorem 5 Let $\kappa = (G, M, I)$ be a formal context. τ is a topology on G and $L(\kappa')$ is a base of τ, where $\tau = \{A; \exists B \subseteq L(\kappa')(A = \bigcup_{X \in B} X)\}$.

Proof. (1) $\bigcup_{X \in B} X = G$ is trivial. (2) Let $X_1, X_2 \in L(\kappa')$. For every $x \in X_1 \bigcap X_2$, we have $\{x\} \subseteq X_1$, $\{x\}^{\uparrow} \supseteq X_1^{\uparrow}$ and hence $\{x\}^{\uparrow\downarrow} \subseteq X_1^{\uparrow\downarrow}$. Similarly, $\{x\}^{\uparrow\downarrow} \subseteq X_2^{\uparrow\downarrow}$ and thus $\{x\}^{\uparrow\downarrow} \subseteq X_1^{\uparrow\downarrow} \bigcap X_2^{\uparrow\downarrow}$. By $X_1^{\uparrow\downarrow} \in L(\kappa')$ we know that $\tau = \{A; \exists B \subseteq L(\kappa')(A = \bigcup_{X \in B} X)\}$ is a topology and $L(\kappa')$ is a topological base of τ.

IV. ROUGHNESS OF UNCERTAIN CONCEPTS

Given a formal context $\kappa = (G, M, I)$. We note that κ can be thought as an information system. In the framework of rough set, Each $Y \subseteq M$ induces an indiscernibility relation R_Y on G which is defined as $(x, y) \in R_Y$ if and only if $(x, g) \in I \Leftrightarrow (y, g) \in I$ for each $g \in Y$. We suppose that (X, Y) is a formal concept in κ. By $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$, it follows that X is an equivalence class with respect to the equivalence relation R_Y.

In what follows, we consider covering rough set model [17], which is a generalization of Pawlak's rough set [18]. For a formal context $\kappa = (G, M, I)$, by Theorem 2, $L(\kappa')$ is a cover of G and $(G, L(\kappa'))$ forms a covering approximation space. In this section, we discuss the roughness of the concept in $\kappa = (G, M, I)$ from rough set point of view.

For each $x \in G$, $N(x) = \bigcap\{X; X \in L(\kappa'), x \in X\}$ is called the neighborhood of x with respect to $L(\kappa')$ [19].

Theorem 6 Let $\kappa = (G, M, I)$ be a formal context. Then we have $N(x) = \{x\}^{\uparrow\downarrow}$ for every $x \in G$.

Proof. Let $x \in G$. We have $\{x\}^{\uparrow\downarrow} \in L(\kappa')$. Furthermore, if $X \in L(\kappa')$ such that $x \in X$, then $\{x\}^{\uparrow} \supseteq X^{\uparrow}$ and hence $\{x\}^{\uparrow\downarrow} \subseteq X^{\uparrow\downarrow} = X$. It follows that $N(x) = \bigcap\{X; X \in L(\kappa'), x \in X\} = \{x\}^{\uparrow\downarrow}$.

Based on covering rough approximation operators [19], we propose the following definition.

Definition 4 Let $\kappa = (G, M, I)$ be a formal context and $X \subseteq G$. The covering based lower approximation $\underline{R}(X)$ and upper approximation $\overline{R}(X)$ of X is defined respectively as follows:

$$\underline{R}(X) = \{x \in G; N(x) \subseteq X\} = \{x \in G; \{x\}^{\uparrow\downarrow} \subseteq X\} \quad (7)$$

$$\overline{R}(X) = \{x \in G; N(x) \bigcap X \neq \emptyset\} = \{x \in G; \{x\}^{\uparrow\downarrow} \bigcap X \neq \emptyset\} \quad (8)$$

The accuracy degree $Ad_C(X)$ of X is defined as $Ad_C(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|}$.

The properties of operators $\underline{R}(X)$ and $\overline{R}(X)$ can be discussed based on covering rough set theory. For example, we have

Theorem 7 Let $\kappa = (G, M, I)$ be a formal context and $X \subseteq G$. Then

$$\underline{R}(X) = \bigcup\{\{x\}^{\uparrow\downarrow}; x \in G, \{x\}^{\uparrow\downarrow} \subseteq X\}.$$

Proof. For every $x \in G$, by $x \in \{x\}^{\uparrow\downarrow}$, it follows that

$$\underline{R}(X) = \bigcup\{\{x\}^{\uparrow\downarrow}; x \in G, \{x\}^{\uparrow\downarrow} \subseteq X\}.$$

Conversely, if $y \in \bigcup\{\{x\}^{\uparrow\downarrow}; x \in G, \{x\}^{\uparrow\downarrow} \subseteq X\}$, then there exists $x \in G$ such that $y \in \{x\}^{\uparrow\downarrow}$ and $\{x\}^{\uparrow\downarrow} \subseteq X$. It follows that $\{y\} \in \{x\}^{\uparrow\downarrow}$ and hence $\{y\}^{\uparrow\downarrow} \subseteq \{x\}^{\uparrow\downarrow\uparrow\downarrow} = \{x\}^{\uparrow\downarrow} \subseteq X$. By the definition, we have $y \in \underline{R}(X)$. Thus $\underline{R}(X) = \bigcup\{\{x\}^{\uparrow\downarrow}; x \in G, \{x\}^{\uparrow\downarrow} \subseteq X\}$ as required.

Example 3 We consider the formal context $\kappa = (G, M, I)$ in Example 1. Let $X = \{3,4,6\}$. By routine computation, we have $\{1\}^{\uparrow\downarrow} = \{1,2,3\}$, $\{2\}^{\uparrow\downarrow} = \{2,3\}$, $\{3\}^{\uparrow\downarrow} = \{3\}$, $\{4\}^{\uparrow\downarrow} = \{4\}$, $\{5\}^{\uparrow\downarrow} = \{5,6\}$, $\{6\}^{\uparrow\downarrow} = \{6\}$, $\{7\}^{\uparrow\downarrow} = \{7\}$, $\{8\}^{\uparrow\downarrow} = \{6,8\}$.It follows that

$$\underline{R}(X) = \{x \in G, \{x\}^{\uparrow\downarrow} \subseteq X\} = \{3,4,6\},$$

$$\overline{R}(X) = \{x \in G; N(x) \cap X \neq \varnothing\} = \{1,2,3,4,5,6,8\}.$$

Consequently, $Ad_C(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|} = \frac{3}{7}$. We note that $\underline{R}(X)$ is a uncertain concept, $\overline{R}(X)$ is an exact concept. If we let $Y = \{4,5\}$, then $\underline{R}(X) = \{4\}$, $\overline{R}(X) = \{4,5\}$. It follows that $\underline{R}(X)$ is an exact concept, whereas $\overline{R}(X)$ is an uncertain concept.

V. CONCLUDING REMARKS

In this paper, based on approximation operators in FCA and covering rough set model, we discuss the uncertain concept and its accuracy degree in a formal context and present approaches to characterizing the accuracy degree of uncertain concept. Based on this paper, we can further probe the applications of FCA in the fields such as pattern recognition, data analysis and decision making. Furthermore, the uncertainty based on other covering rough approximation operators is an important issue to be addressed.

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