The Unbounded Kink Wave of Dullin-Gottwald-Holm Equation

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***Abstract*—The Dullin-Gottwald-Holm equation are investigated by using the dynamical system theory. The Dullin-Gottwald-Holm equation has unbounded kink waves, their existent conditions are found, and exact representations of unbounded kink wave solutions are given.**

Keywords-Dullin-Gottwald-Holm equation; traveling wave equation; unbounded kink wave

I. INTRODUCTION

Meng Q etc [1] has studied the periodic wave structure and properties of Dullin-Gottwald-Holm equation

$$u_t + c_0 u_x + 3uu_x - \alpha^2(u_{xxt} + uu_{xxx} + 2uu_{xx} + \gamma u_{xxx}) = 0 \quad (1)$$

and has given the parameters expression of periodic wave solution.In this paper used the bifurcation method[1-4] of dynamical systems to study DGH equation.The unbounded kink waves of DGH equation are found, and given the function expression of the unbounded kink waves. Under some parameter conditions, in the $\xi - u$ plane, the eight plane curve graphs of unbounded kink waves are painted by using the software Mathematica 7.

II. TRAVELING WAVE EQUATION

Let$\xi = x - ct, u(x,t) = \varphi(\xi)$, where c is the wave speed, then the equation (1) becomes:

$$\gamma\varphi''' - \alpha^2[(\varphi - c)\varphi''' + 2\varphi'\varphi''] + (3\varphi + c_0 - c)\varphi' = 0 \quad (2)$$

Integrating this equation (2) once and taking the integration constant is zero.Let $\beta = c + \frac{\gamma}{\alpha^2}$, we have

$$\alpha^2(\varphi - \beta)\varphi'' = \frac{3}{2}\varphi^2 + (c_0 - c)\varphi - \frac{1}{2}\alpha^2(\varphi')^2 \quad (3)$$

Equation (3) is called the traveling wave equation.

In (3) multiply that by $2\varphi'$ and integrating it, we have

$$\alpha^2(\varphi - \beta)(\varphi')^2 - \varphi^3 - (c_0 - c)\varphi^2 = h \quad (4)$$

where h is integration constant.

Let

$$f_0(\varphi) = -\varphi^3 - (c_0 - c)\varphi^2 \quad (5)$$

We have

$$f_0'(\varphi) = -3\varphi^2 - 2(c_0 - c)\varphi. \quad (6)$$

Clearly,when $c - c_0 > 0, \varphi_+ = 0$,is the minimum point of $f_0'(\varphi)$, $f_0(0) = 0$ is the minimum; $\varphi_- = \frac{2}{3}(c - c_0)$ is the maximum point of $f_0(\varphi)$, $f_0\left(\frac{2}{3}(c - c_0)\right) = \frac{4}{27}(c - c_0)^3$ is maximum; when $c - c_0 < 0, \varphi_+ = 0$ is the maximum point of $f_0(\varphi)$, $f_0(\varphi) = 0$ is maximum; $\varphi_- = \frac{2}{3}(c - c_0)$ is the minimum point of $f_0'(\varphi)$, $f_0\left(\frac{2}{3}(c - c_0)\right) = \frac{4}{27}(c - c_0)^3$ is the minimum.

Let

$$f(\varphi) = -f_0(\varphi) + h \quad (7)$$

We have $h = 0$,

$$f(\varphi) = \varphi^2[\varphi - (c - c_0)] \quad (8)$$

when $h = \frac{4}{27}(c - c_0)^3$,

$$f(\varphi) = [\varphi - \tfrac{2}{3}(c - c_0)]^2\left[\varphi + \tfrac{1}{3}(c - c_0)\right] \quad (9)$$

and（4）become:

$$(\varphi')^2 = \frac{f(\varphi)}{\alpha^2(\varphi - \beta)} \quad (10)$$

III. UNBOUNDED KINK WAVE

Substituting（8）in (10), we have:

$$\frac{d\varphi}{d\xi} = \pm\sqrt{\frac{\varphi^2[\varphi - (c - c_0)]}{\alpha^2(\varphi - \beta)}} \quad (11)$$

When $c - c_0 > 0, \beta > 0, \varphi < 0$, taking $\varphi(0) = \varphi_0 < 0$ as the initial value,and integrating(11), we have:

$$\int_{\varphi_0}^{\varphi} \frac{\beta - s}{-s\sqrt{[s - (c - c_0)](s - \beta)}} ds = \pm\frac{1}{|\alpha|}\int_0^{\xi} ds \quad (12)$$

Set $a_1 = \frac{1}{2}(c - c_0 + \beta), a_2 = \sqrt{\beta(c - c_0)}$.Calculating (12), we obtain the following results respectively:

$$\left\{-\varphi - \sqrt{[\varphi - (c - c_0)](\varphi - \beta)} + a_1\right\}\{-\frac{\sqrt{[\varphi - (c - c_0)](\varphi - \beta)} + a_2}{\varphi} + \frac{a_1}{a_2}\}^{\frac{\beta}{a_2}} = P_0 e^{\frac{\xi}{|\alpha|}} \quad (13)$$

$$\left\{-\varphi - \sqrt{[\varphi - (c - c_0)](\varphi - \beta)} + a_1\right\}\{-\frac{\sqrt{[\varphi - (c - c_0)](\varphi - \beta)} + a_2}{\varphi} + \frac{a_1}{a_2}\}^{\frac{\beta}{a_2}} = P_0 e^{-\frac{\xi}{|\alpha|}} \quad (14)$$

where

$$P_0=\left\{-\varphi_0-\sqrt{[\varphi_0-(c-c_0)](\varphi_0-\beta)}+a_1\right\}\left\{-\frac{\sqrt{[\varphi_0-(c-c_0)](\varphi_0-\beta)}+a_2}{\varphi_0}+\frac{a_1}{a_2}\right\}^{\frac{\beta}{a_2}}$$

When $c-c_0<0,\beta<0,\varphi>0$, taking $\varphi(0)=\varphi_0>0$ as the initial value, and integrating(11),we have:

$$\int_{\varphi_0}^{\varphi}\frac{s-\beta}{s\sqrt{[s-(c-c_0)](s-\beta)}}ds=\pm\frac{1}{|\alpha|}\int_0^{\xi}ds \qquad (15)$$

Set $a_1=\frac{1}{2}(c-c_0+\beta), a_2=\sqrt{\beta(c-c_0)}$ .Calculating (15), we obtain the following results respectively:

$$\left\{\varphi+\sqrt{[\varphi-(c-c_0)](\varphi-\beta)}-a_1\right\}\left\{-\frac{\sqrt{[\varphi-(c-c_0)](\varphi-\beta)}+a_2}{\varphi}-\frac{a_1}{a_2}\right\}^{\frac{\beta}{a_2}}=P_0e^{\frac{\xi}{|\alpha|}} \qquad (16)$$

$$\left\{\varphi+\sqrt{[\varphi-(c-c_0)](\varphi-\beta)}-a_1\right\}\left\{\frac{\sqrt{[\varphi-(c-c_0)](\varphi-\beta)}+a_2}{\varphi}-\frac{a_1}{a_2}\right\}^{\frac{\beta}{a_2}}=P_0e^{-\frac{\xi}{|\alpha|}} \qquad (17)$$

where

$$P_0=\left\{\varphi_0+\sqrt{[\varphi_0-(c-c_0)](\varphi_0-\beta)}+a_1\right\}\left\{\frac{\sqrt{[\varphi_0-(c-c_0)](\varphi_0-\beta)}+a_2}{\varphi_0}-\frac{a_1}{a_2}\right\}^{\frac{\beta}{a_2}}$$

Substituting（9）in (10),we have

$$\frac{d\varphi}{d\xi}=\pm\sqrt{\frac{[\varphi-\frac{2}{3}(c-c_0)]^2[\varphi+\frac{1}{3}(c-c_0)]}{\alpha^2(\varphi-\beta)}} \qquad (18)$$

When $c-c_0>0,\beta<\frac{2}{3}(c-c_0),\varphi>\frac{2}{3}(c-c_0)$,taking $\varphi(0)=\varphi_0>\frac{2}{3}(c-c_0)$ as initial value，integrating(11),we have:

$$\int_{\varphi_0}^{\varphi}\frac{s-\beta}{[s-\frac{2}{3}(c-c_0)]\sqrt{[s+\frac{1}{3}(c-c_0)](s-\beta)}}ds=\pm\frac{1}{|\alpha|}\int_0^{\xi}ds \qquad (19)$$

Set

$b_1=\frac{1}{6}(c-c_0-3\beta), b_2=\sqrt{(c-c_0)[\frac{2}{3}(c-c_0)-\beta]}, b_3=\frac{5}{3}(c-c_0)-\beta$ Calculating (19), we obtain the following results respectively:

$$\left\{\varphi+\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_1\right\}\left\{\frac{\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_2}{\varphi-\frac{2}{3}(c-c_0)}+\frac{b_3}{2b_2}\right\}^{-\frac{\frac{2}{3}(c-c_0)-\beta}{b_2}}=P_0e^{\frac{\xi}{|\alpha|}} \qquad (20)$$

$$\left\{\varphi+\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_1\right\}\left\{\frac{\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_2}{\varphi-\frac{2}{3}(c-c_0)}+\frac{b_3}{2b_2}\right\}^{-\frac{\frac{2}{3}(c-c_0)-\beta}{b_2}}$$

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Where

$$P_0=\left\{\varphi_0+\sqrt{\left[\varphi_0+\frac{1}{3}(c-c_0)\right](\varphi_0-\beta)}+b_1\right\}\left\{\frac{\sqrt{\left[\varphi_0+\frac{1}{3}(c-c_0)\right](\varphi_0-\beta)}+b_2}{\varphi_0-\frac{2}{3}(c-c_0)}+\frac{b_3}{2b_2}\right\}^{-\frac{\frac{2}{3}(c-c_0)-\beta}{b_2}}$$

When $c-c_0<0,\beta>\frac{2}{3}(c-c_0),\varphi<\frac{2}{3}(c-c_0)$ taking $\varphi(0)=\varphi_0<\frac{2}{3}(c-c_0)$ as initial value，integrating(18),we have:

$$\int_{\varphi_0}^{\varphi}\frac{-(s-\beta)}{-[s-\frac{2}{3}(c-c_0)]\sqrt{[s+\frac{1}{3}(c-c_0)](s-\beta)}}ds=\pm\frac{1}{|\alpha|} \qquad (22)$$

Set

$b_1=\frac{1}{6}(c-c_0-3\beta), b_2=\sqrt{(c-c_0)[\frac{2}{3}(c-c_0)-\beta]}, b_3=\frac{5}{3}(c-c_0)-\beta$. Calculating(19), we obtain the following results respectively:

$$\left\{\varphi+\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_1\right\}\left\{\frac{\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_2}{\frac{2}{3}(c-c_0)-\varphi}-\frac{b_3}{2b_2}\right\}^{-\frac{\frac{2}{3}(c-c_0)-\beta}{b_2}}=P_0e^{\frac{\xi}{|\alpha|}} \qquad (23)$$

$$\left\{\varphi+\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_1\right\}\left\{\frac{\sqrt{\left[\varphi+\frac{1}{3}(c-c_0)\right](\varphi-\beta)}+b_2}{\frac{2}{3}(c-c_0)-\varphi}-\frac{b_3}{2b_2}\right\}^{-\frac{\frac{2}{3}(c-c_0)-\beta}{b_2}}=P_0e^{-\frac{\xi}{|\alpha|}} \qquad (24)$$

Where

$$P_0=\left\{\varphi_0+\sqrt{\left[\varphi_0+\frac{1}{3}(c-c_0)\right](\varphi_0-\beta)}+b_1\right\}\left\{\frac{\sqrt{\left[\varphi_0+\frac{1}{3}(c-c_0)\right](\varphi_0-\beta)}+b_2}{\frac{2}{3}(c-c_0)-\varphi_0}-\frac{b_3}{2b_2}\right\}^{-\frac{\frac{2}{3}(c-c_0)-\beta}{b_2}}$$

When $c=c_0$, (11) and (18) are changed to:

$$\frac{d\varphi}{d\xi} = \pm\sqrt{\frac{\varphi^3}{\alpha^2(\varphi-\beta)}} \quad (25)$$

When $\beta > 0, \varphi < 0$, taking $\varphi(0) = \varphi_0 < 0$ as initial value, integrating (25),we have:

$$\int_{\varphi_0}^{\varphi} \frac{\beta - s}{-s\sqrt{s(s-\beta)}} ds = \pm\frac{1}{|\alpha|}\int_0^{\xi} ds \quad (26)$$

Calculating (26), we obtain the following result respectively:

$$\ln\frac{-\varphi-\sqrt{\varphi(\varphi-\beta)}+\frac{\beta}{2}}{-\varphi_0-\sqrt{\varphi_0(\varphi_0-\beta)}+\frac{\beta}{2}} - \frac{2\sqrt{\varphi(\varphi-\beta)}}{\varphi} + \frac{2\sqrt{\varphi_0(\varphi_0-\beta)}}{\varphi_0} = \frac{\xi}{|\alpha|} \quad (27)$$

$$\ln\frac{-\varphi-\sqrt{\varphi(\varphi-\beta)}+\frac{\beta}{2}}{-\varphi_0-\sqrt{\varphi_0(\varphi_0-\beta)}+\frac{\beta}{2}} - \frac{2\sqrt{\varphi(\varphi-\beta)}}{\varphi} + \frac{2\sqrt{\varphi_0(\varphi_0-\beta)}}{\varphi_0} = -\frac{\xi}{|\alpha|} \quad (28)$$

When $\beta < 0, \varphi > 0$, taking $\varphi(0) = \varphi_0 > 0$ as initial value, integrating（25）,we have:

$$\int_{\Phi_0}^{\varphi} \frac{s-\beta}{s\sqrt{s(s-\beta)}} ds = \pm\frac{1}{|\alpha|}\int_0^{\xi} ds \quad (29)$$

Calculating (29), we obtain the following result respectively:

$$\ln\frac{\varphi+\sqrt{\varphi(\varphi-\beta)}-\frac{\beta}{2}}{\varphi_0+\sqrt{\varphi_0(\varphi_0-\beta)}-\frac{\beta}{2}} - \frac{2\sqrt{\varphi(\varphi-\beta)}}{\varphi} + \frac{2\sqrt{\varphi_0(\varphi_0-\beta)}}{\varphi_0} = \frac{\xi}{|\alpha|} \quad (30)$$

$$\ln\frac{\varphi+\sqrt{\varphi(\varphi-\beta)}-\frac{\beta}{2}}{\varphi_0+\sqrt{\varphi_0(\varphi_0-\beta)}-\frac{\beta}{2}} - \frac{2\sqrt{\varphi(\varphi-\beta)}}{\varphi} + \frac{2\sqrt{\varphi_0(\varphi_0-\beta)}}{\varphi_0} = -\frac{\xi}{|\alpha|} \quad (31)$$

When $c = c_0, \beta = 0$,(25) is changed to:

$$\frac{d\varphi}{d\xi} = \pm\sqrt{\frac{\varphi^2}{\alpha^2}} \quad (32)$$

When $\varphi < 0$,taking $\varphi(0) = \varphi_0 < 0$ as initial value，integrating (32),we have:

$$\int_{\varphi_0}^{\varphi} -\frac{1}{s} ds = \pm\frac{1}{|\alpha|}\int_0^{\xi} ds \quad (33)$$

Calculating (33),we obain the following results respectively:

$$\varphi = \varphi_0 e^{-\frac{\xi}{|\alpha|}} \quad (34)$$

$$\varphi = \varphi_0 e^{\frac{\xi}{|\alpha|}} \quad (35)$$

When $\varphi > 0$,taking $\varphi(0) = \varphi_0 > 0$ as initial value,integrating (32),we have

$$\int_{\varphi_0}^{\varphi} \frac{1}{s} ds = \pm\frac{1}{|\alpha|} \quad (36)$$

Calculating (36), we obtain the following results respectively:

$$\varphi = \varphi_0 e^{\frac{\xi}{|\alpha|}} \quad (37)$$

$$\varphi = \varphi_0 e^{-\frac{\xi}{|\alpha|}} \quad (38)$$

By derivation of the above have the following theorem:

Theorem

If $c - c_0 > 0, \beta > 0$ and $\varphi < 0$,then the equation (1) has two unbounded kink wave solutions and they have implicit expression (13) and (14);

If $c - c_0 < 0, \beta < 0$ and $\varphi > 0$,then the equation (1) has two unbounded kink wave solutions and they have implicit expression (16) and (17);

If $c - c_0 > 0, \beta < \frac{2}{3}(c - c_0)$ and $\varphi > \frac{2}{3}(c - c_0)$,then the equation (1) has two unbounded kink wave solutions and they have implicit expression (20) and (21);

If $c - c_0 < 0, \beta > \frac{2}{3}(c - c_0)$ and $\varphi < \frac{2}{3}(c - c_0)$， then the equation (1) has two unbounded kink wave solutions and they have implicit expression (23) and (24);

If $c = c_0, \beta > 0$ and $\varphi < 0$，then the equation (1) has two unbounded kink wave solutions and they have implicit expression (27) and (28);

If $c = c_0, \beta < 0$ and $\varphi > 0$，then the equation (1) has two unbounded kink wave solutions and they have implicit expression (30) and (31);

If $c = c_0, \beta = 0$ and $\varphi < 0$，then the equation (1) has two unbounded kink wave solutions and they have implicit expression (34) and (35);

If $c = c_0, \beta = 0$ and $\varphi > 0$，then the equation (1) has two unbounded kink wave solutions and they have implicit expression (37) and (38);

Example 1 Taking $\alpha = 1, \gamma = 0, c_0 = 2, c = 3$， we have $\beta = 3$. Taking $\varphi_0 = -4$, substituting these data into (13) and (14), using mathematical software Mathematica 7.0, respectively ,we draw two unbounded kink wave plane curves graph as Fig.1a.

Example 2 Taking $\alpha = 1, \gamma = 0, c_0 = 2, c = -3$， we have $\beta = -3$. Taking $\varphi_0 = 4$,substituting these data into (16) and (17), using mathematical software Mathematica 7.0, respectively ,we draw two unbounded kink wave plane curves graph as Fig.1b.

Example 3 Taking $\alpha = 1, \gamma = -5, c_0 = 2, c = 3$， we have $\beta = -2$. Taking $\varphi_0 = 4$, substituting these data into (20) and (21), using mathematical software Mathematica 7.0, respectively ,we draw two unbounded kink wave plane curves graph as Fig.1c.

Example 4 Taking $\alpha = 1, \gamma = 1, c_0 = 2, c = 1$， we have$\beta = 2$. Taking $\varphi_0 = -4$, substituting these data into (23) and (24), using mathematical software Mathematica 7.0, respectively ,we draw two unbounded kink wave plane curves graph as Figure.1d.

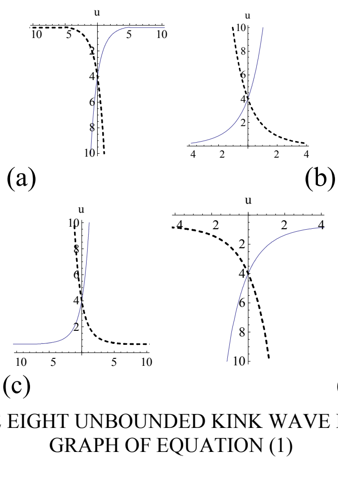

FIGURE I. THE EIGHT UNBOUNDED KINK WAVE PLANE CURVE GRAPH OF EQUATION (1)

(a)$\alpha = 1, \gamma = 0, c_0 = 2, c = 3, \varphi_0 = -4.$
(b)$\alpha = 1, \gamma = 0, c_0 = 2, c = -3, \varphi_0 = 4.$
(c)$\alpha = 1, \gamma = -5, c_0 = 2, c = 3, \varphi_0 = 4.$
(d)$\alpha = 1, \gamma = 1, c_0 = 2, c = 1, \varphi_0 = -4.$

IV. CONCLUSION

Has been the kink wave conclusion is more of a bounded kink wave. In this paper, we study the DGH equation to find the existence of unbounded kink wave. Using dynamical systems theory and differential calculus theory to find the conditions for the existence of unbounded kink wave,and calculate the expression of unbounded kink.Numerical simulation show the plane graph of unbounded kink wave with the exponential function characteristics.

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