Specific Solutions of a Class of Second Order Difference Equation with Boundary Conditions

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***Abstract*—Difference equation is a kind of important tool to study the rule of natural phenomena. In this paper, we discuss several specific solutions of a class of second order difference equation with boundary conditions.**

Keywords-difference equation; second order; boundary condition; specific solution

I. Introduction

Difference equations is a kind of important tool to study the rule of natural phenomena, such as, physical problems arising in a wide variety of applications. Cheng and Cho [3] investigated the following second order difference equations

$$\Delta^2 x(k-1) + p(k)x(k) = 0\,,$$

where p(k) is a real valued function defined on a set of the natural numbers.

Motivated by the results given in [1, 2, 3, 4, 5], in this paper, we discuss speciffc solutions of the following second order difference equations for $k \in \{1,2,\cdots N\}$,

$$\Delta^2 x(k-1) + p(k)x(k) = 0, \tag{1}$$

satisfying

$$x(0) = 0, x(N+1) = 0,$$

or

$$x(0) + \sigma x(1) = 0, x(N+1) + \lambda x(N) = 0,$$

or

$$x(0) + \sigma x(1) = 0, x(2m) + \lambda x(2m) = 0, x(4m+1) + \theta x(4m) = 0.$$

II. Main Results

Throughout this paper, let n, m be natural numbers,

$$I_{n,m} = \{n, n+1, \cdots, m\}$$

Proposition 1. Let $N = 2m+1$,

$$p(k) = \begin{cases} 0, & k \in I_{0,m}, \\ 2/(m+1), & k = m+1, \\ 0, & k \in I_{m+2,N}, \end{cases} \tag{2}$$

and

$$x(k) = \begin{cases} k, & k \in I_{0,m+1}, \\ 2m-k+2, & k \in I_{m+1,N+1}. \end{cases} \tag{3}$$

Then (3) is the specific solution of second order difference equation (1).

Proof. From (2) and (3), we have

$$\begin{aligned} &\frac{\Delta^2 x(m)}{x(m+1)} \\ &= \frac{x(m+2) - 2x(m+1) + x(m)}{x(m+1)} \\ &= \frac{m - 2(m+1) + m}{m+1} \\ &= \frac{-2}{m+1} = -p(m+1) \end{aligned} \tag{4}$$

Since $\Delta^2 x(k-1)/x(k) = 0 = -p(k)$ for $k \in I_{1,m} \bigcup k \in I_{m+2,N}$, (3) is a specific solution of (1) with $x(0) = 0, x(N+1) = 0$.

Proposition 2. Let $N = 2m$,

$$p(k) = \begin{cases} 0, & k \in I_{0,m}, \\ (2m+1)/m(m+1), & k = m+1, \\ 0, & k \in I_{m+2,N}, \end{cases} \tag{5}$$

and

$$x(k) = \begin{cases} k, & k \in I_{0,m+1}, \\ (m+1)(2m-k+1)/m, & k \in I_{m+1,N+1}. \end{cases} \tag{6}$$

Then (6) is the specific solution of (1).

Proof. From (5) and (6), we obtain that

$$\begin{aligned} &\frac{\Delta^2 x(m)}{x(m+1)} \\ &= \frac{x(m+2) - 2x(m+1) + x(m)}{x(m+1)} \\ &= \frac{(m^2-1)/m - 2(m+1) + m}{m+1} \\ &= \frac{m^2 - 1 - 2m(m+1) + m^2}{m(m+1)} \\ &= -\frac{2m+1}{m+1} = -p(m+1). \end{aligned} \tag{7}$$

Since $\Delta^2 x(k-1)/x(k)=0=-p(k)$ for $k\in I_{1,m}\bigcup k\in I_{m+2,N,}$ (6) is a specific solution of (1) such that

$x(0)=0, x(N+1)=0$.

Proposition 3. Let $N=2m+1$,

$$p(k)=\begin{cases}0, & k\in I_{0,m,}\\ \dfrac{N+1+N\lambda+N\sigma+2m\lambda\sigma}{(m+1+m\sigma)(m+1+m\lambda)}, & k=m+1,\\ 0, & k\in I_{m+2,N,}\end{cases} \tag{8}$$

and

$$x(k)=\begin{cases}1+(1+\sigma)(k-1), & k\in I_{0,m+1,}\\ \dfrac{(m+1+m\sigma)[2m+1+1/(1+\lambda)-k]}{m+1/(1+\lambda)}, & k\in I_{m+1,N+1.}\end{cases} \tag{9}$$

Then (9) is the specific solution of (1) with

$$x(0)+\sigma x(1)=0, x(N+1)=\lambda x(N)=0.$$

Proof. From (8) and (9), we conclude

$$\begin{aligned}&\frac{\Delta^2 x(m)}{x(m+1)}\\ &=\frac{x(m+2)-2x(m+1)+x(m)}{x(m+1)}\\ &=\frac{x(m+2)-x(m+1)-(x(m+1)-x(m))}{x(m+1)}\\ &=\frac{1+\sigma}{m+1+m\sigma}+\frac{m+1+m\sigma}{(m+1/(1+\lambda))(m+1+m\sigma)}\\ &=\frac{1+\sigma}{m+1+m\sigma}+\frac{1+\lambda}{m+1+m\lambda}\\ &=\frac{(1+\sigma)(m+1+m\lambda)+(1+\lambda)(m+1+m\sigma)}{(m+1+m\sigma)(m+1+m\lambda)}\\ &=\frac{2m+2+(2m+1)\lambda+(2m+1)\sigma+2m\lambda\sigma}{(m+1+m\sigma)(m+1+m\lambda)}\\ &=\frac{N+1+N\lambda+N\sigma+2m\lambda\sigma}{(m+1+m\sigma)(m+1+m\lambda)}=-p(m+1)\end{aligned} \tag{10}$$

Since

$\Delta^2 x(k-1)/x(k)=0=-p(k)$ for $k\in I_{1,m}\bigcup k\in I_{m+2,N,}$

(9) is a specific solution of (1) with

$$x(0)+\sigma x(1)=0.\quad x(N+1)+\lambda x(N)=0.$$

Proposition 4. Let $N=2m$,

$$p(k)=\begin{cases}0, & k\in I_{0,m-1,}\\ \dfrac{N+1+N\lambda+N\sigma+(N-1)\lambda\sigma}{(m+(m-1)\sigma)(m+1+m\lambda)} & k=m\\ 0, & k\in I_{m+1,N,}\end{cases} \tag{11}$$

and

$$x(k)=\begin{cases}1+(1+\sigma)(k-1) & k\in I_{0,m,}\\ \dfrac{(m(1+\sigma)-\sigma)(2m+1/(1+\lambda)-k)}{m+1/(1+\lambda)}, & k\in I_{m,N+1,}\end{cases} \tag{12}$$

Then (12) is the solution of (1) with

$$x(0)+\sigma x(1)=0, x(N+1)+\lambda x(N)=0$$

Proof. Using (11) and (12), we have

$$\begin{aligned}&-\frac{\Delta^2 x(m-1)}{x(m)}\\ &=-\frac{x(m+1)-2x(m)+(m-1)}{x(m)}\\ &=-\frac{x(m+1)-x(m)-(x(m)-x(m-1))}{x(m)}\\ &=\frac{m(1+\sigma)-\sigma}{(m+1/(1+\lambda))(m(1+\sigma)-\sigma)}+\frac{1+\sigma}{m(1+\sigma)-\sigma}\\ &=\frac{1+\sigma}{m+(m-1)\sigma}+\frac{1+\lambda}{m+1+m\lambda}\\ &=\frac{(1+\sigma)(m+1+m\lambda)+(1+\lambda)(m+(m-1)\sigma)}{(m+(m-1)\sigma)(m+1+m\lambda)}\\ &=\frac{2m+1+2m\lambda+2m\sigma+(2m-1)\lambda\sigma}{(m+(m-1)\sigma)(m+1m\lambda)}\\ &=p(m),\end{aligned} \tag{13}$$

$\Delta^2 x(k-1)/x(k)=0=-p(k)$ for $k\in I_{1,m}\bigcup k\in I_{m+2,N,}$ (12) is a specific solution of (1) with

$$x(0)+\sigma x(1)=0, x(N+1)+\lambda x(N)=0.$$

Proposition 5. Let

$$p(k)=\begin{cases}0, & k\in I_{0,m-1,}\\ \dfrac{2m+1+2m\lambda+2m\sigma+(2m-1)\lambda\sigma}{(m+(m+1)\sigma)(m+1+m\lambda)}, & k=m\\ 0, & k\in I_{m+1,3m-1,}\\ \dfrac{2m+2m\lambda+2m\theta+2m\lambda\theta+\lambda-\theta}{(m+m\theta+1)(m+m\lambda-1)}, & k=3m,\\ 0, & k\in I_{3m+1,4m},\end{cases} \tag{14}$$

and

$$x(k)=\begin{cases}1+(1+\sigma)(k-1), & k\in I_{0,m},\\ \dfrac{(m(1+\sigma)-\sigma)(2m+1/(1+\lambda)-k)}{m+1/(1+\lambda)}, & k\in I_{m,3m},\\ \dfrac{(m(1+\sigma)-\sigma)(1/(1+\lambda)-m)}{(m+1/(1+\lambda))(m+1/(1+\theta))}\times(4m+\dfrac{1}{1+\theta}-k), & k\in I_{3m,4m+1.}\end{cases} \tag{15}$$

Then (15) is the solution of (1) with

$$x(0)+\sigma x(1)=0, x(2m+1)+\lambda x(2m)=0, x(4m+1)+\theta x(4m)=0$$

.

Proof. Using (14) and (15), we have

$$-\frac{\Delta^2 x(m-1)}{x(m)}=\frac{2m+1+2m\lambda+2m\sigma+(2m-1)\lambda\sigma}{(m+(m-1)\sigma)(m+1+m\lambda)}=p(m), \tag{16}$$

$$\begin{aligned}&-\frac{\Delta^2 x(3m-1)}{x(3m)}\\ &=-\frac{x(3m+1)-2x(3m)+x(3m-1)}{x(3m)}\end{aligned}$$

$$= -\frac{x(3m+1)-x(3m)-(x(3m)-x(3m-1))}{x(3m)}$$

$$= \frac{1+\theta}{m+m\theta+1} + \frac{1+\lambda}{m+m\lambda-1}$$

$$= \frac{(1+\theta)(m+m\lambda-1)+(1+\lambda)(m+m\theta+1)}{(m+m\theta+1)(m+m\lambda-1)}$$

$$= \frac{2m+2m\lambda+2m\theta+2m\lambda\theta+\lambda-\theta}{(m+m\theta+1)(m+m\lambda-1)}$$

$$= p(3m), \qquad (17)$$

Since

$$\Delta^2 x(k-1)/x(k) = 0 = -p(k)$$

for $k \in I_{1,m-1} \bigcup I_{m+1,3m-1} \bigcup k \in I_{3m+1,4m}$, (15) is a specific solution of (1) with

$$x(0)+\sigma x(1)=0, x(2m+1)+\lambda x(2m)=0, x(4m+1)+\theta x(4m)=0$$

III. SUMMARY

Difference equation is a kind of important tool to study the rule of natural phenomena. In this paper, we discuss several specific solutions of a class of second order difference equation with boundary conditions.

ACKNOWLEDGEMENTS

This research was supported by Scientific Research Foundation of the Education Department of Guangxi Autonomous Region of China (No. 201010LX473).

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