Nonlinear Weakly Singular Iterated Integral Inequality

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***Abstract*—In this paper, we establish a class of new nonlinear weakly singular integral inequality, which is solved by adopting novel analysis techniques, such as: differential and integration, inverse function, and explicit bounds for the unknown functions are given clearly.**

Keywords-integral inequality; weakly singular integral kernel; iterated integrals; analysis technique; estimation

I. INTRODUCTION

In 2011, Abdeldaim et al. [1] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t) \le u_0 + \int_{t_0}^{t} f(s)u(s)[u(s) + \int_{t_0}^{s} h(\tau)[u(\tau) + \int_{t_0}^{\tau} g(\xi)u(\xi)d\xi]d\tau]ds \quad (1)$$

To avoid the shortcoming of these results, Medved [2] presented a new method to discuss nonlinear singular integral inequalities of Henry type and their Bihari version as follows:

$$u(t) \le a(t) + \int_0^t (t-s)^{\beta-1} f(s)w(u(s))ds \quad (2)$$

and the estimates of solutions are given, respectively.

Motivated by the results given in [1-5], in this paper, we discuss a new retarded nonlinear integral inequality

$$u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta_1-1} f(s)w_1(u(s))[u(s) + \int_{t_0}^{s} (s-\tau)^{\beta_2-1} g(\tau)w_2(u(\tau))[u(\tau) + \int_{t_0}^{\tau} (\tau-\xi)^{\beta_3-1} h(\xi)w_3(u(\xi))d\xi]d\tau]ds \quad (3)$$

for all $t \in [t_0, J)$. The inequality (3) consist of iterated integral, and weakly singular integral kernel be involved in each layer. Under several practical assumptions, the inequality is solved through rigorous analysis, and explicit bounds for the unknown functions are given clearly. Moreover, an example is presented to show the usefulness of our results.

II. MAIN RESULT

Throughout this paper, R denotes the set of real numbers, $R_+=[0,+\infty]$, $I=[t_0,J)$; $C^1(M,S)$ denotes the class of continuously differentiable functions defined on set *M* with range in the set *S*, C(*M*,*S*) denotes the class of continuously functions defined on set *M* with range in the set *S*, $\beta'(t)$ denotes the derived function of a function $\beta'(t)$.

For convenience, before giving our main results, we cite some useful lemmas and definitions in the discussion of our proof as follows:

Definition 1. ([2])

Let q >0 be a real number and 0 <T<∞. We say that a function $w: R^+ \to R$ satisfies a condition (q), if

$$e^{-qt}[w(u)]^q \le R(t)w(e^{-qt}u^q), \forall u \in R^+, t \in [0,T), \quad (4)$$

where R(t) is a continuous, nonnegative function.

Lemma 1. ([2])

Let t_0, t, $\beta \in R^+$. Let $\beta > 1/2$, then $2\beta - 1 > 0$ and

$$\int_{t_0}^{t} (t-s)^{2\beta-2} e^{2s} ds \le \frac{2e^{2t}}{4^\beta}\Gamma(2\beta-1) \quad (5)$$

where $2\beta - 1 > 0$, $\Gamma(\beta) := \int_0^\infty \tau^{\beta-1}e^{-\tau}d\tau$ is the gamma function.

Lemma 2. (Discrete Jensen inequality [3])

Let $A_1, A_2, \cdots, A_n$ be nonnegative real numbers, r>1 is a real numbers, and n is a natural number. Then

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r). \quad (6)$$

Theorem 1.

Suppose that $b(t), f(t), g(t) \in C(I, R_+)$, $\phi_i \in C(R_+, R_+)(i = 1,2,3)$ with $\phi_i(0) = 0$, $\phi_i(u) > 0$, for u>0. Let $\beta_i > 1/2\,(i = 1,2)$, w_1, w_2 satisfy the condition (4) with q=2. If $u(t)$ satisfies (3), then

$$u(t) \le e^t\{\Omega_1^{-1}\{\Omega_2^{-1}\{\Omega_3^{-1}\{\widetilde{A}(t)\}\}\}\}^{-1/2}, \forall t \in [t_0, T_1), \quad (7)$$

where

$$\widetilde{A}(t) = \Omega_3[\Omega_2(\Omega_1(A(t)) + \int_{t_0}^{t} K_3\widetilde{h}(s)ds) + \int_{t_0}^{t} K_2\widetilde{g}(s)ds] + \int_{t_0}^{t} K_1\widetilde{f}(s)ds,$$

$$\Omega_1(z) = \int_{z_0}^{z} \frac{ds}{w_3(s)}, z_0 > 0, z \in (z_0, +\infty), \quad (8)$$

$$\Omega_2(z) = \int_{z_0}^{z} \frac{w_3(\Omega_1^{-1}(s))ds}{w_2(\Omega_1^{-1}(s))\Omega_1^{-1}(s)}, z_0 > 0, z \in (z_0, +\infty), \quad (9)$$

$$\Omega_3(z) = \int_{z_0}^{z} \frac{w_2(\Omega_1^{-1}(\Omega_2^{-1}(s)))ds}{w_1(\Omega_1^{-1}(\Omega_2^{-1}(s)))}, z_0 > 0, z \in (z_0, +\infty), \quad (10)$$

and $\Omega_1^{-1}, \Omega_2^{-1}, \Omega_3^{-1}$ are inverse functions of $\Omega_1, \Omega_2, \Omega_3$, respectively,

$A(t)=2a^2(t), K_1=\frac{2\Gamma(2\beta_1-1)}{4^{\beta_1-1}}, K_2=\frac{\Gamma(2\beta_2-1)}{4^{\beta_2-1}}, K_3=\frac{2\Gamma(2\beta_3-1)}{4^{\beta_2}},$

$\tilde{f}(s)=f^2(s)R_1(s)e^{2s}, \tilde{g}(\tau)=g^2(\tau)R_2(\tau)e^{2\tau}, \tilde{h}(\xi)=h^2(\xi)R_3(\xi).$

and $R_1(t), R_2(t), R_3(t)$ are defined by (4), T_1 is the largest real number such that

$$T_1=Max\{t\in I, \tilde{A}(t)\in Dom\Omega_3^{-1}, \Omega_3^{-1}\{\tilde{A}(t)\}\in Dom\Omega_2^{-1}, \Omega_2^{-1}\{\Omega_3^{-1}\{\tilde{A}(t)\}\}\in Dom\Omega_1^{-1}\}.$$

Proof. Using the Cauchy-Schwarz inequality, we obtain from (3) that

$$\begin{aligned} u(t) &\le a(t)+\int_{t_0}^{t}(t-s)^{\beta_1-1}e^s f(s)e^{-s}w_1(u(s))[u(s) \\ &\quad +\int_{t_0}^{s}(s-\tau)^{\beta_2-1}e^{\tau}g(\tau)e^{-\tau}w_2(u(\tau))[u(\tau) \\ &\quad +\int_{t_0}^{\tau}(\tau-\xi)^{\beta_3-1}e^{\xi}h(\xi)e^{-\xi}w_3(u(\xi))d\xi]d\tau]ds \\ &\le a(t)+[\int_{t_0}^{t}(t-s)^{2\beta_1-2}e^{2s}ds]^{\frac{1}{2}}[\int_{t_0}^{t}f^2(s)e^{-2s}w_1^2(u(s))[u(s) \\ &\quad +[\int_{t_0}^{s}(s-\tau)^{2\beta_2-2}e^{2\tau}d\tau]^{\frac{1}{2}}[\int_{t_0}^{s}e^{-2\tau}g^2(\tau)w_2^2(u(\tau))[u(\tau) \\ &\quad +[\int_{t_0}^{\tau}(\tau-\xi)^{2\beta_3-2}e^{2\xi}d\xi]^{\frac{1}{2}} \\ &\quad \times[\int_{t_0}^{\tau}e^{-2\xi}h^2(\xi)w_3^2(u(\xi))d\xi]^{\frac{1}{2}}]^2d\tau]^{\frac{1}{2}}]^2ds]^{\frac{1}{2}}, \forall t\in I. \end{aligned} \tag{11}$$

Using discrete Jensen inequality (3) with n = 2, r = 2, we obtain from (11) that

$$\begin{aligned} u^2(t) &\le 2a^2(t)+2[\int_{t_0}^{t}(t-s)^{2\beta_1-2}e^{2s}ds][\int_{t_0}^{t}f^2(s)e^{-2s}w_1^2(u(s))[2u^2(s) \\ &\quad +2[\int_{t_0}^{s}(s-\tau)^{2\beta_2-2}e^{2\tau}d\tau][\int_{t_0}^{s}e^{-2\tau}g^2(\tau)w_2^2(u(\tau))[2u^2(\tau) \\ &\quad +2[\int_{t_0}^{\tau}(\tau-\xi)^{2\beta_3-2}e^{2\xi}d\xi][\int_{t_0}^{\tau}e^{-2\xi}h^2(\xi)w_3^2(u(\xi))d\xi]]d\tau]]ds], \end{aligned} \tag{12}$$

for all $t\in I$. Using the condition (4) in definition 1 and (5) in Lemma 1, from (12) we obtain that

$$\begin{aligned} u^2(t) &\le 2a^2(t)+2[\frac{2e^{2t}}{4^{\beta_1}}\Gamma(2\beta_1-1)][\int_{t_0}^{t}f^2(s)R_1(s)w_1(e^{-2s}u^2(s))[2u^2(s) \\ &\quad +2[\frac{2e^{2s}}{4^{\beta_2}}\Gamma(2\beta_2-1)][\int_{t_0}^{s}g^2(\tau)R_2(\tau)w_2(e^{-2\tau}u^2(\tau))[2u^2(\tau) \\ &\quad +2[\frac{2e^{2\tau}}{4^{\beta_3}}\Gamma(2\beta_3-1)] \\ &\quad \times[\int_{t_0}^{\tau}h^2(\xi)R_3(\xi)w_3(e^{-2\xi}u^2(\xi))d\xi]]d\tau]]ds] \\ &\le 2a^2(t)+4[\frac{2e^{2t}}{4^{\beta_1}}\Gamma(2\beta_1-1)] \\ &\quad \times[\int_{t_0}^{t}f^2(s)R_1(s)e^{2s}w_1(e^{-2s}u^2(s))[u^2(s)e^{-2s} \\ &\quad +2[\frac{2}{4^{\beta_2}}\Gamma(2\beta_2-1)] \\ &\quad \times[\int_{t_0}^{s}g^2(\tau)R_2(\tau)e^{2\tau}w_2(e^{-2\tau}u^2(\tau))[2u^2(\tau)e^{-2\tau}+[\frac{2}{4^{\beta_3}}\Gamma(2\beta_3-1)] \\ &\quad \times[\int_{t_0}^{\tau}h^2(\xi)R_3(\xi)w_3(e^{-2\xi}u^2(\xi))d\xi]]d\tau]]ds], \quad \forall t\in I. \end{aligned} \tag{13}$$

Let $v(t)=u^2(t)e^{-2t}$, we have from (13)

$$\begin{aligned} v(t) &\le A(t)+K_1\int_{t_0}^{t}\tilde{f}(s)w_1(v(s))[v(s)+K_2\int_{t_0}^{s}\tilde{g}(\tau)w_2(v(\tau))[v(\tau) \\ &\quad +K_3\int_{t_0}^{\tau}\tilde{h}(\xi)w_3(v(\xi))d\xi]d\tau]ds, \forall t\in I, \end{aligned} \tag{14}$$

where

$A(t)=2a^2(t)$, $K_1=\frac{2\Gamma(2\beta_1-1)}{4^{\beta_1-1}}$, $K_2=\frac{\Gamma(2\beta_2-1)}{4^{\beta_2-1}}$, $K_3=\frac{2\Gamma(2\beta_3-1)}{4^{\beta_3}}$,

$\tilde{f}(s)=f^2(s)R_1(s)e^{2s}$, $\tilde{g}(\tau)=g^2(\tau)R_2(\tau)e^{2\tau}$, $\tilde{h}(\xi)=h^2(\xi)R_3(\xi)$.

Let $z(t)$ denote the function on the right-hand side of (14), which is a positive and nondecreasing function on I. From (14), we have

$$v(t)\le z(t), A(t)\le z(t), \forall t\in I \tag{15}$$

Differentiating $z(t)$ with respect to t, using (15) we have

$$\begin{aligned} z'(t) &\le A'(t)+K_1\tilde{f}(t)w_1(z(t))[z(t)+K_2\int_{t_0}^{t}\tilde{g}(\tau)w_2(z(\tau))[z(\tau) \\ &\quad +K_3\int_{t_0}^{\tau}\tilde{h}(\xi)w_3(z(\xi))d\xi]d\tau] \\ &= A'(t)+K_1\tilde{f}(t)w_1(z(t))z_1(t), \forall t\in I, \end{aligned} \tag{16}$$

Where

$$z_1(t)=z(t)+K_2\int_{t_0}^{t}\tilde{g}(\tau)w_2(z(\tau))[z(\tau)+K_3\int_{t_0}^{\tau}\tilde{h}(\xi)w_3(z(\xi))d\xi]d\tau. \tag{17}$$

Hence, z_1 is a positive and nondecreasing function on I,

$$z_1(t_0)=z(t_0),\ z(t)\le z_1(t),\ \forall t\in I. \tag{18}$$

Differentiating $z_1(t)$ with respect to t, using (16) and (18) we have

$$\begin{aligned} z_1'(t) &\le A'(t)+K_1\tilde{f}(t)w_1(z_1(t))z_1(t) \\ &\quad +K_2\tilde{g}(t)w_2(z_1(t))[z_1(t)+K_3\int_{t_0}^{t}\tilde{h}(\xi)w_3(z_1(\xi))d\xi] \\ &= A'(t)+K_1\tilde{f}(t)w_1(z_1(t))z_1(t)+K_2\tilde{g}(t)w_2(z_1(t))z_2(t), \end{aligned} \tag{19}$$

for all $t\in I$, and

$$z_2(t)=z_1(t)+K_3\int_{t_0}^{t}\tilde{h}(\xi)w_3(z_1(\xi))d\xi. \tag{20}$$

Hence, z_2 is a positive and nondecreasing function on I,

$$z_2(t_0)=z(t_0), z_1(t)\le z_2(t), \forall t\in I. \tag{21}$$

Differentiating $z_2(t)$ with respect to t, using (19) and (21) we have

$$\begin{aligned} z_2'(t) &= z_1'(t)+K_3\tilde{h}(t)w_3(z_1(t)) \\ &\le A'(t)+K_1\tilde{f}(t)w_1(z_2(t))z_2(t)+K_2\tilde{g}(t)w_2(z_1(t))z_2(t) \\ &\quad +K_2\tilde{g}(t)w_2(z_1(t))z_2(t)+K_3\tilde{h}(t)w_3(z_2(t)), \\ &\quad \forall t\in I. \end{aligned} \tag{22}$$

Since w_3, z_2 are nondecreasing functions on I, we have from (22)

$$\frac{z_2'(t)}{w_3(z_2(t))} \le \frac{A'(t)}{w_3(z_2(t))} + K_1\tilde{f}(t)z_2(t)\frac{w_1(z_2(t))}{w_3(z_2(t))} + K_2\tilde{g}(t)z_2(t)\frac{w_2(z_2(t))}{w_3(z_2(t))} + K_3\tilde{h}(t)$$
$$\le \frac{A'(t)}{w_3(A(t))} + K_1\tilde{f}(t)z_2(t)\frac{w_1(z_2(t))}{w_3(z_2(t))} + K_2\tilde{g}(t)z_2(t)\frac{w_2(z_2(t))}{w_3(z_2(t))} + K_3\tilde{h}(t), \quad \forall t \in I. \tag{23}$$

where we used the relation $A(t) \le z(t) \le z_1(t) \le z_2(t)$ .Integrating both sides of the above inequality from t_0 to t , we obtain

$$\Omega_1(z_2(t)) \le \Omega_1(A(t)) + \int_{t_0}^{t} K_1\tilde{f}(s)z_2(s)\frac{w_1(z_2(s))}{w_3(z_2(s))}ds + \int_{t_0}^{t} K_2\tilde{g}(s)z_2(s)\frac{w_2(z_2(s))}{w_3(z_2(s))}ds + \int_{t_0}^{t} K_3\tilde{h}(s)ds$$
$$\le \Omega_1(A(T)) + \int_{t_0}^{T} K_3\tilde{h}(s)ds + \int_{t_0}^{t} K_1\tilde{f}(s)z_2(s)\frac{w_1(z_2(s))}{w_3(z_2(s))}ds + \int_{t_0}^{t} K_2\tilde{g}(s)z_2(s)\frac{w_2(z_2(s))}{w_3(z_2(s))}ds \tag{24}$$

for $t_0 \le t \le T \le T_1$, T is chosen arbitrarily, where Ω_1 is defined by (8). Let $z_3(t)$ denote the function on the right-hand side of (24), which is a positive and nondecreasing function on $[t_0, T]$. From (24), we have

$$z_3(t_0) = \Omega_1(A(T)) + \int_{t_0}^{T} K_3\tilde{h}(s)ds, \quad z_2(t) \le \Omega_1^{-1}(z_3(t)). \tag{25}$$

Differentiating $z_3(t)$ with respect tot, using (25) we have

$$z_3'(t) = K_1\tilde{f}(t)z_2(t)\frac{w_1(z_2(t))}{w_3(z_2(t))} + K_2\tilde{g}(t)z_2(t)\frac{w_2(z_2(t))}{w_3(z_2(t))}$$
$$\le K_1\tilde{f}(t)\Omega_1^{-1}(z_3(t))\frac{w_1(\Omega_1^{-1}(z_3(t)))}{w_3(\Omega_1^{-1}(z_3(t)))} + K_2\tilde{g}(t)\Omega_1^{-1}(z_3(t))\frac{w_2(\Omega_1^{-1}(z_3(t)))}{w_3(\Omega_1^{-1}(z_3(t)))}, \tag{26}$$

for all $t \in [t_0, T]$ .Since w_2/w_3 , Ω_1^{-1} , z_3 are positive and nondecreasing functions on $[t_0, T]$,from (26) we have

$$\frac{z_3'(t)w_3(\Omega_1^{-1}(z_3(t)))}{w_2(\Omega_1^{-1}(z_3(t)))\Omega_1^{-1}(z_3(t))} \le K_2\tilde{g}(t) + K_1\tilde{f}(t)\frac{w_1(\Omega_1^{-1}(z_3(t)))}{w_2(\Omega_1^{-1}(z_3(t)))}, \tag{27}$$

for all $t \in [t_0, T]$,Integrating both sides of the above inequality from t_0 to t , we obtain that

$$\Omega_2(z_3(t)) \le \Omega_2(z_3(t_0)) + \int_{t_0}^{t} K_2\tilde{g}(s)ds + \int_{t_0}^{t} K_1\tilde{f}(s)\frac{w_1(\Omega_1^{-1}(z_3(s)))}{w_2(\Omega_1^{-1}(z_3(s)))}ds$$
$$\le \Omega_2(z_3(t_0)) + \int_{t_0}^{T} K_2\tilde{g}(s)ds + \int_{t_0}^{t} K_1\tilde{f}(s)\frac{w_1(\Omega_1^{-1}(z_3(s)))}{w_2(\Omega_1^{-1}(z_3(s)))}ds \tag{28}$$

for all $t \in [t_0, T]$,where Ω_2 is defined by (9). Let $z_4(t)$ denote the function on the right-hand side of (28), which is a positive and nondecreasing function on $[t_0, T]$.From (28), we have

$$z_4(t_0) = \Omega_2(z_3(t_0)) + \int_{t_0}^{T} K_2\tilde{g}(s)ds = \Omega_2(\Omega_1(A(T)) + \int_{t_0}^{T} K_3\tilde{h}(s)ds) + \int_{t_0}^{T} K_2\tilde{g}(s)ds, \tag{29}$$

$$z_3(t) \le \Omega_2^{-1}(z_4(t)), \forall t \in [t_0, T]. \tag{30}$$

Differentiating $z_4(t)$ with respect tot, using (30) we have

$$z_4'(t) = K_1\tilde{f}(t)\frac{w_1(\Omega_1^{-1}(z_3(t)))}{w_2(\Omega_1^{-1}(z_3(t)))} \le K_1\tilde{f}(t)\frac{w_1(\Omega_1^{-1}(\Omega_2^{-1}(z_4(t))))}{w_2(\Omega_1^{-1}(\Omega_2^{-1}(z_4(t))))}, \ \forall t \in [t_0, T]. \tag{31}$$

From (31), we have

$$\frac{z_4'(t)w_2(\Omega_1^{-1}(\Omega_2^{-1}(z_4(t))))}{w_1(\Omega_1^{-1}(\Omega_2^{-1}(z_4(t))))} \le K_1\tilde{f}(t), \ \forall t \in [t_0, T]. \tag{32}$$

Integrating both sides of the above inequality from t_0 to t, we obtain that

$$\Omega_3(z_4(t)) \le \Omega_3(z_4(t_0)) + \int_{t_0}^{t} K_1\tilde{f}(s)ds, \ \forall t \in [t_0, T], \tag{33}$$

where Ω_3 is defined by (10).

From (15), (18), (21), (25), (30) and (33), we have

$$v(t) \le z(t) \le z_1(t) \le z_2(t) \le \Omega_1^{-1}(z_3(t)) \le \Omega_1^{-1}(\Omega_2^{-1}(z_4(t)))$$
$$\le \Omega_1^{-1}\{\Omega_2^{-1}\{\Omega_3^{-1}\{\Omega_3[\Omega_2(\Omega_1(A(T)) + \int_{t_0}^{T} K_3\tilde{h}(s)ds) + \int_{t_0}^{T} K_2\tilde{g}(s)ds] + \int_{t_0}^{t} K_1\tilde{f}(s)ds\}\}\}, \forall t \in [t_0, T]. \tag{34}$$

Since T is chosen arbitrarily, we have

$$v(t) \le \Omega_1^{-1}\{\Omega_2^{-1}\{\Omega_3^{-1}\{\Omega_3[\Omega_2(\Omega_1(A(t)) + \int_{t_0}^{t} K_3\tilde{h}(s)ds) + \int_{t_0}^{t} K_2\tilde{g}(s)ds] + \int_{t_0}^{t} K_1\tilde{f}(s)ds\}\}\}, \tag{35}$$

for all $t \in [t_0, T_1]$. In view of $v(t) = u^2(t)e^{-2t}$, we can obtain (7).

III. SUMMARY

In this paper, we establish a class of new nonlinear weakly singular integral inequality. By adopting novel analysis techniques, we have obtained explicit bounds for the unknown functions in the inequality.

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