Implementation of Infeasible Kernel-Based Interior-Point Methods for Linearly Constrained Convex Optimization

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***Abstract*—In this paper, we present the implementation of infeasible kernel-based primal-dual interior-point methods for linearly constrained convex optimization. Numerical results are provided to demonstrate the efficiency of the algorithms.**

Keywords-interior-point methods; kernel function; linearly constrained convex optimization; primal-dual methods; polynomial complexity

I INTRODUCTION

In this paper, we consider the linearly constrained convex optimization (LCCO) problem

$$(P)\quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax=b,\ x\geq 0 \end{array}$$

And its dual problem

$$(D)\quad \begin{array}{ll} \text{maximize} & b^T y+f(x)-\nabla f(x)^T x \\ \text{subject to} & A^T y+s-\nabla f(x)=0,\ s\geq 0 \end{array}$$

Where $A\in R^{m\times n}$ with rank (A)= m , $b\in R^m$, $f(x):R^n\to R$ is a convex and twice continuously differentiable, $\nabla f(x)$ is the gradient vector of $f(x)$. It is obvious that LCCO reduces to convex quadratic optimization (CQO) if $f(x)=c^T x+\frac{1}{2}x^T Qx$, that is to say, LCCO is the generalization of CQO, which contains linear optimization (LO) (i.e., $Q=0$) as a special case. Since the groundbreaking paper of Karmarkar, many researchers have proposed and analyzed various interior-point methods (IPMs) for LO and a large amount of results have been reported. For a further survey we refer to recent monograph on the subject [4, 8].

In the past few years, CQO has received considerable attention from researchers because of the close connection between LO and CQO. Monteiro and Alder [6] presented a primal-dual path-following interior-point algorithm for CQO. Yu et al. [9] considered a polynomial predictor-corrector interior-point algorithm for CQO. Cai et al. [3] proposed a class of primal-dual interior-point algorithms for CQO based on a finite barrier and obtained the currently best known iteration bound for large-and small-update methods.

There are a variety of solution approaches for the LCCO problem which have been studied intensively. Among them, the IPMs gained many attentions than others methods. Zhu [11] proposed a path-following interior-point algorithm for a class of convex programming problems, including LCCO as a special case. Monteiro [5] studied the global convergence of a large class of primal-dual interior-point algorithms for LCCO. It is generally agreed that primal-dual path-following methods are most efficient from a computational point of view. However, there is a gap between the practical behaviour of the IPMs and the theoretical performance results. The so-called small-update IPMs enjoy the best known worst-case iteration bound $O(\sqrt{n}\log(n/\varepsilon))$ but their performance in computational practice is poor. In practice, however, the so-called large-update IPMs are much more efficient than small-update IPMs but with relatively weak theoretical result $O(n\log(n/\varepsilon))$.

Recently, Peng et al. [7] introduced the so-called self-regular barrier functions and presented a class of primal-dual IPMs for LO based on self-regular proximities. The currently best known iteration bounds for large- and small-update methods are established, which almost close the gap between the theoretical iteration bounds for large- and small-update methods. Bai et al. [2] introduced a large class of eligible kernel functions, which is fairly general and includes the classical logarithmic function and the self-regular functions, as well as many non-self-regular functions as special cases. The best known iteration bounds for LO obtained are as good as the ones in [7] for appropriate choices of the eligible kernel functions.

The purpose of the paper is to present the implementation of infeasible primal-dual IPMs for LCCO based on the eligible kernel functions. Some preliminary numerical results are reported to show the efficiency of the proposed algorithms.

The outline of the paper is as follows. In Section 2, we briefly recall the eligible kernel functions and the corresponding barrier functions. The framework of the infeasible kernel-based primal-dual IPMs for LCCO is presented in Section 3. Numerical results are provided in Section 4. Finally, some concluding remarks are made in Section 5.

II THE ELIGIBLE KERNEL (BARRIER) FUNCTIONS

In this section, we briefly recall the eligible kernel functions and the corresponding barrier functions that are used in the algorithms. For more details on the kernel functions we refer to [2, 7].

A univariate $\psi:(0,+\infty)\to[0,+\infty)$ is called a kernel function [7] if it satisfies

$$\psi'(1)=\psi(1)=0,\ \psi''(t)>0,\ \lim_{t\to 0^+}\psi(t)=\lim_{t\to+\infty}\psi(t)=+\infty. \quad (1)$$

One can easily verify that $\psi(t)$ is strictly convex and minimal at $t=1$, with $\psi(1)=0$.

We recall the so-called eligible kernel function [2], i.e., the kernel function satisfies four of the following five conditions, namely the first and the last three conditions.

$$\begin{aligned} t\psi''(t)+\psi'(t)>0,\ t<1,\\ t\psi''(t)-\psi'(t)>0,\ t>1,\\ \psi'''(t)<0,\ t>0,\\ 2\psi''(t)^2-\psi'(t)\psi'''(t)>0,\ t<1,\\ \psi''(t)\psi'(\beta t)-\beta\psi'(t)\psi''(\beta t)>0,\ t>1,\beta>1. \end{aligned} \quad (2)$$

It should be pointed out that the first four conditions are logically independent, and that the fifth condition is a consequence of the second condition and the third condition. Since the second condition is much simpler to check than the fifth condition, in many cases it is easy to know that $\psi(t)$ is eligible if it satisfies the first four conditions. Some well-known eligible kernel functions are presented below (see, e.g., [2, 3]).

TABLE I: EIGHT ELIGIBLE KERNEL FUNCTIONS.

$\varphi_1(t)=\frac{t^2-1}{2}-\log t;$	$\varphi_2(t)=\frac{t^{p+1}-1}{p+1}-\log t,\ p\in[0,1];$
$\varphi_3(t)=\frac{1}{2}\left(t-\frac{1}{t}\right)^2;$	$\varphi_4(t)=\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q-1},\ q>1;$
$\varphi_5(t)=t-\frac{1}{t}-2;$	$\varphi_6(t)=\frac{t^{p+1}-1}{p+1}+\frac{t^{1-q}-1}{q-1},\ p\in[0,1],q>1;$
$\varphi_7(t)=\frac{t^2-1}{2}+\frac{e^{q(1/t-1)}-q}{q},\ q\ge 1;$	$\varphi_8(t)=\frac{t^2-1}{2}+\int_1^t e^{q(1/\xi-1)}\mathrm{d}\xi,\ q\ge 1.$

Corresponding to the kernel function $\psi(t)$, we define the barrier function $\Psi(v):R^n_{++}\to R_+$ as follows

$$\Psi(x,s,\mu):=\Psi(v):=\sum_{i=1}^{n}\psi(v_i).$$

One can easily verify that $\Psi(v)$ is a strictly convex function and attains minimal values at $v=e$ and $\Psi(e)=0$. Then we have

$$\nabla\Psi(v)=0\Leftrightarrow\Psi(v)=0\Leftrightarrow v=e.$$

III INFEASIBLE KERNEL-BASED PRIMAL-DUAL IPMS FOR LCCO

A. The Central Path

Throughout the paper, we assume that the LCCO problem satisfies the interior-point condition (IPC), i.e., there exists (x^0,y^0,s^0) such that

$$Ax^0=b,\ x^0>0,\ A^Ty^0+s^0-\nabla f(x^0)=0,\ s^0>0.$$

It is well-known that the IPC can be assumed without loss of generality. In fact we may, and will assume that $x^0=s^0=e$. For this and some other properties mentioned below (see, e.g., [5, 10]).

The Karush-Kuhn-Tucker optimality conditions for the problems are given as follows

$$\left.\begin{aligned} Ax=b,\ \ x>0,\\ A^Ty+s-\nabla f(x)=0,\ \ s>0,\\ xs=0. \end{aligned}\right\} \quad (3)$$

The basic idea of the primal-dual interior-point algorithm is to replace the third equation in (3), the so-called complementarity condition for LCCO, by the parameterized equation $xs=\mu e$, with $\mu>0$. Thus we consider the following system

$$\left.\begin{aligned} Ax=b,\ \ x>0,\\ A^Ty+s-\nabla f(x)=0,\ \ s>0,\\ xs=\mu e. \end{aligned}\right\} \quad (4)$$

The parameterized system (4) has a unique solution for each $\mu>0$. This solution is denoted as $(x(\mu),y(\mu),s(\mu))$ and we call $x(\mu)$ the μ-center of (P) and $(y(\mu),s(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all the positive real numbers) gives a homotopy path, which is called the central path of LCCO. If $\mu\to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields an ε-approximate solution of LCCO (see, e.g., [5, 10]).

B. The New Search Directions

IPMs follow the central path approximately and find an approximate solution of the underlying problems (P) and (D) as μ go to zero. Applying Newton's method, we have

$$\left.\begin{aligned} A\Delta x=b-Ax,\\ A^T\Delta y+\Delta s-\nabla^2 f(x)\Delta x=\nabla f(x)-A^Ty-s,\\ s\Delta x+x\Delta s=\mu e-xs. \end{aligned}\right\} \quad (5)$$

For further uses we introduce the scaled vector as follows

$$v:=\sqrt{\frac{xs}{\mu}}.$$

One can conclude that the pair (x,s) coincides with the μ-center $(x(\mu),s(\mu))$ if and only if $v=e$. The classical logarithmic barrier function is defined by

$$\Psi_c(v):=\sum_{i=1}^{n}\left(\frac{v_i^2-1}{2}-\log v_i\right),\quad v\in R^n.$$

Let $\nabla\Psi_c(v)$ denotes the gradient of the classical logarithmic barrier function $\Psi_c(v)$. We have $\mu e - xs = -\mu v \nabla\Psi_c(v)$. This implies that the system (5) is equivalent to the following system

$$\left.\begin{aligned} A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s - \nabla^2 f(x)\Delta x &= \nabla f(x) - A^T y - s, \\ s\Delta x + x\Delta s &= -\mu v \nabla\Psi_c(v). \end{aligned}\right\} \qquad (6)$$

In this paper, we replace the right-hand side of the third equation in (6) by the gradient $\nabla\Psi(v)$ of the barrier function $\Psi(v)$, where $\varphi(t)$ is any eligible kernel function. Then we consider the following system

$$\left.\begin{aligned} A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s - \nabla^2 f(x)\Delta x &= \nabla f(x) - A^T y - s, \\ s\Delta x + x\Delta s &= -\mu v \nabla\Psi(v). \end{aligned}\right\} \qquad (7)$$

The new search direction $(\Delta x, \Delta y, \Delta s)$ is computed by solving the system (7). If $(x, y, s) \neq (x(\mu), y(\mu), s(\mu))$, then $(\Delta x, \Delta y, \Delta s)$ is nonzero. By taking a step along the search direction, with the step size α defined by some line search rules, one constructs a new triple (x^+, y^+, s^+) according to

$$x^+ = x + \alpha\Delta x,\ y^+ = y + \alpha\Delta y,\ s^+ = s + \alpha\Delta s.$$

The generic form of infeasible kernel-based interior-point algorithm for LCCO is shown below.

Algorithm 1 Primal-Dual Interior-Point Algorithm for LCCO

Step 0 Input a threshold parameter $\tau \geq 1$, an accuracy parameter $\varepsilon > 0$, a fixed barrier update parameter $0 < \theta < 1$, a strictly feasible (x^0, y^0, s^0) and $\mu^0 = 1$ such that $\Psi(x^0, s^0; \mu^0) < \tau$.
Set $x := x^0, y := y^0, s := s^0, \mu = \mu^0.$

Step 1 If $n\mu < \varepsilon$, stop, (x, y, s) is an optimal solution; otherwise, update $\mu := (1-\theta)\mu$, go to Step 2.

Step 2 If $\Psi(x, s; \mu) < \tau$, go back to Step 1; otherwise, go to Step 3.

Step 3 Solve system (7) to obtain $(\Delta x, \Delta y, \Delta s)$, go to Step 4.

Step 4 Choose a default step size α, go to Step 5.

Step 5
Update $x^+ := x + \alpha\Delta x,\ y^+ := y + \alpha\Delta y,\ s^+ := s + \alpha\Delta s$, go back to Step 2.

IV NUMERICAL RESULTS

In this section, we report some numerical results. The maximum allowable step sizes uses in our experiments. Let

$$\alpha_P^{max} = \frac{1}{max_{i=1,2,\ldots n}\left\{1, -\frac{\Delta x_i}{x_i}\right\}} \quad \text{and} \quad \alpha_D^{max} = \frac{1}{max_{i=1,2,\ldots n}\left\{1, -\frac{\Delta s_i}{s_i}\right\}}.$$

The maximum allowable step sizes are slightly reduced by a fixed factor $0 < \alpha_0 < 1$ (we choose $\alpha_0 = 0.75$) to prevent hitting the boundary, i.e.

$$\alpha_P = \alpha_0\alpha_P^{max} \text{ and } \alpha_D = \alpha_0\alpha_D^{max}.$$

The new iteration point is obtained by

$$x^+ = x + \alpha_P\Delta x,\ y^+ = y + \alpha_D\Delta y,\ s^+ = s + \alpha_D\Delta s.$$

We consider the following two LCCO problems in [1].

Example 1

$$\text{minimize} \quad f(x) = x_1 + 20\left(\frac{x_1}{10}\right)^5 + x_2 + 22.5\left(\frac{x_2}{20}\right)^5 + x_1 + 25\left(\frac{x_3}{30}\right)^5$$

$$\text{subject to} \quad \begin{cases} x_1 + x_2 + x_3 = 100, \\ x_1, x_2, x_3 \geq 0. \end{cases}$$

Without loss of generality, we chose $x = s = (1;1;1)$ and $y = 0$ as the start point. The eligible kernel function used in this paper is $\varphi_6(t)$ with the parameters $p = 1$ and $q = 2$. We choose the fixed barrier update parameter $\theta = 0.5$, the threshold parameter $\tau = 3$, the barrier parameter $\mu = 1$ and the accuracy parameter $\varepsilon = 10^{-6}$.

For our infeasible kernel-based IPMs, we need 14 main iterations to reach our accuracy. An optimal solution of the problem is obtained by

$$x^* = (14.1977; 32.7882; 53.0141),$$

And the optimal value of the problem (P) is equal to 912.6450.

Example 2

TABLE II: THE COEFFICIENTS OF THE OBJECTIVE FUNCTION.

i	1	2	3	4	5	6	7	8
t_i^0	5	8	8	3	3	3	3	8
C_i	15	15	15	10	10	10	10	15

$$\text{minimize} \quad f(x)=\sum_{i=1}^{8} t_i^0\left(x_i+\frac{\alpha C_i}{\beta+1}\left(\frac{x_i}{C_i}\right)^{\beta+1}\right)$$

$$\text{subject to} \begin{cases} x_1 + x_3 = 30, \\ x_1 - x_2 - x_4 + x_5 = 0, \\ x_2 - x_6 + x_7 = 20, \\ -x_3 - x_4 + x_5 + x_8 = 8, \\ x_1, x_2, \dots, x_8 \ge 0. \end{cases}$$

Without loss of generality, we chose $x=s=(1;1;1;1;1;1;1;1)$ and $y=(0;0;0;0)$ as the start point. The eligible kernel function used in this paper is $\varphi_6(t)$ with the parameters $p=1$ and $q=2$. We choose the fixed barrier update parameter $\theta=0.5$, the threshold parameter $\tau=8$, the barrier parameter $\mu=1$ and the accuracy parameter $\varepsilon=10^{-6}$.

For our infeasible kernel-based IPMs, we need 18 main iterations to reach our accuracy. An optimal solution of the problem is obtained by

$$x^* = (16.2528;18.9997;13.7472;0.0000;2.7468;0.0000;1.0003;19.0003)$$

and the optimal value of the problem (P) is equal to 1023.3822.

It should be pointed out that the iteration number of the algorithms depends on the eligible kernel functions and the values of the parameters τ, θ and ε.

V CONCLUDING REMARKS

In this paper, we have implemented infeasible kernel-based primal-dual IPMs for LCCO. Some preliminary numerical results are provided to show the efficiency of the proposed algorithms.

Some interesting topics remain for further research. Firstly, the more details numerical test is an interesting topic to investigate the behaviour of the algorithm so as to compare with other approach. Secondly, the analysis of the convergence of the algorithm is also deserved to be researched. These will be other issue for future research.

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