A Birth-Death Process Model on Collision and Coalescence of Drops in Gas/Drop Flows

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***Abstract*-To analyze the change of droplets number caused by collision and coalescence in gas/droplet two-phase flows, a stochastic model was established, in which the collision probability of two drops with different sizes was derived and the stochastic process theory was introduced to describe the change of drops number when the phenomenon of collision and coalescence takes place. By considering the immigration and emigration and collision of the droplets located at a fixed region, it was found that the number of drops is a birth death process, and the relevant model equation was derived. The stationary distribution and its mean value were obtained by analyzing the number characteristic of the birth-death process. These results are helpful to explain the mass exchange between drop classes in gas/drop two phase flows.**

Keywords- gas/drop two-phase flows; droplet; collision and coalescence; birth-death process

I. INTRODUCTION

The phenomenon of collision and coalescence of drops widely occurs in gas/drop two phase flows, causing the change of the drop's number and sizes[1].In some specific flows, this phenomenon has great effect on the pattern of the two-phase flows, and it is more interested in determining inter-phase mass, momentum and energy exchange. For example, the mass of a rain drop created in cloud increases gradually due to collision and coalescence with other droplets when it falls. When a shock wave travels in a gas/drop mixture, collision and coalescence of droplets with different sizes easy takes place, resulting in the change of the shock wave structure.

It is important for researchers to study quantitatively the change of droplets number caused by collision and coalescence in gas/drop two-phase flows. In such flows, the liquid phase often composes of a large number of multi-size droplets, and the usual approach to deal with is dividing them into many classes, each class having the same size or speed[1].The most simple case deserving mention is that the liquid phase only contain larger and smaller sizes two classes of droplets, each class having the same dynamical behavior, and it is assumed the collision between droplets in each class does not take place, but for inter-class, it takes place. This predigestion is helpful to establish a mathematical model to describe the number change of droplets.

To establish mathematical models by the ideas of probability theory, is an important means to research and solve practical problems[2,3].As an ancient and classical stochastic process, birth and death process has a profound theoretical and practical value, and its application has already extended to the fields of physics, chemistry, and biology[4].In this paper, the idea of stochastic process is introduced to study the change of droplets number caused by collision and coalescence of droplets in gas/droplet two-phase flows. By analyzing these three factors of the immigration, the emigration and the coalescence of droplets, a birth-death process model for the drops number and corresponding unconditional probability model equation was put forward. The solution of model equations can be employed to determine the mass or size change of drops in gas/drop flows.

II. MATHEMATICAL MODEL

A. The Collision Probability

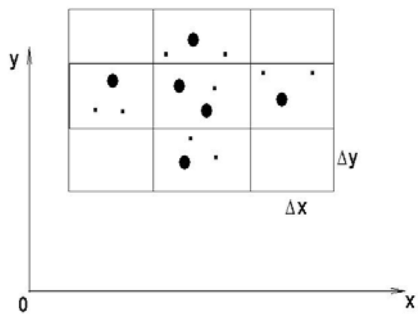

FIGURE I. CHEME OF MESHES.

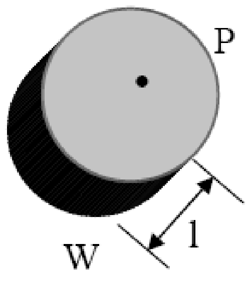

FIGURE II. SCHEME OF A LARGE DROP.

Fig.1. shows the scheme of meshes of gas/droplet two-phase flow field. Let the flow field consist of drops with two sizes: one is called larger drops with number density β, diameter D, having more stable number density distribution. The other is called smaller droplets with number density n,

diameter d. Both of them has random position and velocity $\vec{U}$ and $\vec{V}$ respectively. It is assumed that the relative velocity $\vec{V}-\vec{U}$ with length δ and a angle θ between it and x axis are more significant. After a interval Δt, if the larger drop, indicated by the circle P in fig.2., is moved a distance l, $l=\delta\cdot\Delta t$, in the opposite direction of the vector $\vec{V}-\vec{U}$, then a shade W, with the area $S=D\cdot\delta\cdot\Delta t$,bounded by the two left semicircles of P and two tangential lines, is formed. If there is any small droplet locating at the region W in t moment, then, in $t+\Delta t$ moment, it will be absorbed by the larger drop, in other words, collision and coalescence between drop and droplet takes place.

Taking a rectangular cell Q, with length Δx and height Δy from fig.1. for analysis, the probability that a droplet locates at the shade in t moment is

$$\varepsilon=\frac{\beta\Delta x\cdot\Delta y\cdot D\delta\cdot\Delta t}{\Delta x\cdot\Delta y}=\beta D\delta\cdot\Delta t=\alpha\beta\cdot\Delta t \tag{1}$$

Since the location of each droplet is statistically independent, then the probability that there are $k(k\le m)$ smaller droplets collide with bigger drops is

$$C_m^k\varepsilon^k(1-\varepsilon)^{m-k} \tag{2}$$

Where m is the number of small droplets in Q.

B. Droplet Number Model

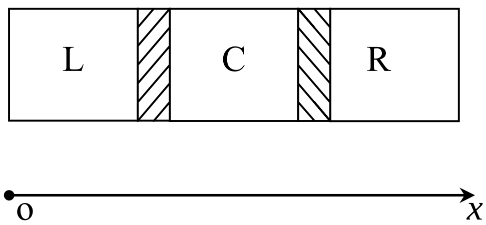

FIGURE III. SCHEME OF ONE DIMENSIONAL MESH FOR DROP MOTION

Assuming that the gases and droplets move in one dimensional mode, and the gas flow's direction is taken as positive direction of X axis(Fig.3.). Many cells with height 1, length Δx are draw in flow region. Gas flow contains a number of small size and large size drops, and their positions is random. The smaller droplets, following the gas, have no slip with larger one in speed. Let the droplets number is x in cell C,xl in cell L in the moment t. After a interval Δt, the number of small droplets in C changes due to droplets immigrating from L and emigrating to R, collision and coalescence. The number of migrating droplets equals that of droplets locating at shade area subjected to L and C respectively. If ul and u respectively indicates gas flow speed in L and C, then the areas of shaded region are ulΔt and uΔt. It is known from the statistical independence of the droplets positions in the cell that, the event that k droplets locate in the shade region, can be interpreted as a result of Bernoulli tests. So, the probability that a droplet immigrate from L cell is

$$C_{x_l}^1\frac{u_l}{\Delta x}\Delta t=x_l\cdot\lambda\cdot\Delta t \tag{3}$$

And a droplet moves to R, is

$$C_x^1\frac{u}{\Delta x}\Delta t=x\cdot\mu\cdot\Delta t \tag{4}$$

The probability that two droplets immigrate is $C_{x_l}^2\lambda^2\Delta t^2+o(\Delta t^2)$, and two droplets emigrating, is $C_x^2\mu^2\Delta t^2+o(\Delta t^2)$ .It is easy seen, the probability that two or more droplets migrate is $o(\Delta t)$. The probability that the number of droplets reduces one due to collision is

$$\alpha\cdot\beta\cdot x\cdot\Delta t \tag{5}$$

So, the probability that adding a droplet to C is $h\Delta t+o(\Delta t)$, $h>0$, reducing a droplet, is $g\Delta t+o(\Delta t)$, $g>0$, and changing in two or more droplets is $o(\Delta t)$ .Let $P_x(t)$ indicates the probability that the droplets number in cell C is x in moment t, then the probability that it keeps x in moment $t+\Delta t$ can be expressed as follows:

$$\begin{aligned}P_x(t+\Delta t)&=P_x(t)\cdot P\,\{\text{no immigrating, no emigrating, no collision}\}\\&+P_x(t)\cdot P\,\{\text{one immigrating, one emigrating, no collision}\}\\&+P_x(t)\cdot P\,\{\text{one immigrating, no emigrating, one reduction by collision}\}\\&+P_{x-1}(t)\cdot P\,\{\text{one immigrating, no emigrating, no collision}\}\\&+P_{x+1}(t)\cdot P\,\{\text{no immigrating, no emigrating, one lose by collision}\}\\&+P_{x+1}(t)\cdot P\,\{\text{no immigrating, one emigrating, no collision}\}\\&+P_{x+1}(t)\cdot P\,\{\text{one immigrating, one emigrating, one lose by collision}\}\\&+P_{x+2}(t)\cdot P\,\{\text{no immigrating, one emigrating, one lose by collision}\}\end{aligned} \tag{6}$$

substituting previous probability formula to eqn(6),have

$$\begin{aligned}P_x(t+\Delta t)&=P_x(t)(1-x_l\cdot\lambda\cdot\Delta t)(1-x\cdot\mu\cdot\Delta t)(1-x\cdot\alpha\cdot\beta\cdot\Delta t)\\&+P_x(t)(x_l\cdot\lambda\cdot\Delta t)(x\cdot\mu\cdot\Delta t)(1-x\cdot\alpha\cdot\beta\cdot\Delta t)\end{aligned}$$

$$+P_x(t)(x_l\cdot\lambda\cdot\Delta t)(1-x\cdot\mu\cdot\Delta t)(x\cdot\alpha\cdot\beta\cdot\Delta t)$$
$$+P_{x-1}(t)(x_l\cdot\lambda\cdot\Delta t)[1-(x-1)\mu\cdot\Delta t][1-\alpha\cdot\beta(x-1)\cdot\Delta t]$$
$$+P_{x+1}(t)(1-x_l\cdot\lambda\cdot\Delta t)[(x+1)\mu\cdot\Delta t][1-\alpha\cdot\beta(x+1)\cdot\Delta t]$$
$$+P_{x+1}(t)(1-x_l\cdot\lambda\cdot\Delta t)[1-(x+1)\mu\cdot\Delta t][\alpha\cdot\beta(x+1)\cdot\Delta t]$$
$$+P_{x+1}(t)(x_l\cdot\lambda\cdot\Delta t)[(x+1)\mu\cdot\Delta t][\alpha\cdot\beta(x+1)\cdot\Delta t]$$
$$+P_{x+2}(t)(1-x_l\cdot\lambda\cdot\Delta t)[(x+2)\mu\cdot\Delta t][\alpha\cdot\beta(x+2)\cdot\Delta t] \quad (7)$$

After simplifying and omitting the high order small quantity of Δt, have

$$P_x(t+\Delta t)-P_x(t)=-P_x(t)(x_l\cdot\lambda+\alpha\cdot\beta\cdot x+x\cdot\mu)\Delta t+P_{x-1}(t)(x_l\cdot\lambda\cdot\Delta t)+P_{x+1}(t)[(x+1)\mu]\Delta t+P_{x+1}(t)[\alpha\cdot\beta(x+1)]\Delta t+o(\Delta t) \quad (8)$$

Divided by Δt in both sides simultaneously, and let $\Delta t\to 0$, then

$$\frac{dP_x(t)}{dt}=x_l\lambda P_{x-1}(t)-(x_l\lambda+\alpha\beta x+\mu x)P_x(t)+(x+1)(\mu+\alpha\beta)P_{x+1}(t)$$

Similarly holds

$$\frac{dP_0(t)}{dt}=-x_l\lambda P_0(t)+(\mu+\alpha\beta)P_1(t) \quad (10)$$

III. SOLUTION OF MODEL EQUATION

In equation (9), (10), if assumes speed ul, u, large-size droplet number density β and the relative speed value δ does not vary with time, then a time homogeneous birth-death process will be obtained. For the convenience of discussion, takes the symbols in the equations consistent with that provided by the literature[4], the equations can be written as

$$\frac{dP_j(t)}{dt}=bP_{j-1}(t)-(ja+b)P_j(t)+(j+1)aP_{j+1}(t) \quad (11)$$
$$\frac{dP_0(t)}{dt}=-bP_0(t)+aP_1(t) \quad (12)$$

Where $b=x_l\lambda=x_l\dfrac{u_l}{\Delta x}$,

$a=\mu+\alpha\beta=\mu+D\cdot\delta\cdot\beta=\dfrac{u}{\Delta x}+D\cdot\delta\cdot\beta$, this is an unconditional probability equation.

Seeking for the steady state solution of birth-death process is important in many practical problems. The stationary distribution depends on judgment of some number characters of this process. The number characters below are important to solve the equation (11) [4]:

$$m_i=\frac{1}{b_i}+\sum_{k=0}^{i-1}\frac{a_ia_{i-1}\cdots a_{i-k}}{b_ib_{i-1}\cdots b_{i-k}b_{i-k-1}}$$
$$=\frac{1}{b}+\frac{1}{b}\sum_{k=0}^{i-1}i(i-1)\cdots(i-k)(\frac{a}{b})^{k+1}$$
$$R=\sum_{i=0}^{\infty}m_i\ge\sum_{i=0}^{\infty}\frac{1}{b}=\infty,$$
$$e_i=\frac{1}{a_i}+\sum_{k=0}^{\infty}\frac{b_ib_{i+1}\cdots b_{i+k}}{a_ia_{i+1}\cdots a_{i+k}a_{i+k+1}}$$

A theorem needs to be introduced as follows[4]

Theorem Assumes $R=\infty$, then (11) exists unique stationary distribution if and only if $e_1<\infty$, and the distribution can be given by

$$v_0=(1+b_0e_1)^{-1}$$
$$v_j=\frac{b_0b_1\cdots b_{j-1}}{a_1a_2\cdots a_j(1+b_0e_1)}(j>0) \quad (13)$$

Where $v_j=\lim_{t\to\infty}P_j(t)$ 。 let $\varepsilon=b/a$, then $e_1=\dfrac{1}{a}\sum_{n=0}^{\infty}\dfrac{1}{(n+1)!}\varepsilon^n$, the convergence of right series can be judged as follows

$$\frac{\frac{1}{(n+2)!}\varepsilon^{n+1}}{\frac{1}{(n+1)!}\varepsilon^n}=\frac{\varepsilon}{n+2}\to 0<1(n\to\infty)$$

Since $\sum_{n=0}^{\infty}\dfrac{1}{(n+1)!}\varepsilon^n$ converges, i.e. $e_1<\infty$, and $e_1=\dfrac{e^{\varepsilon}-1}{a\varepsilon}$ can be deduced. Hence, (11) exists unique stationary distribution

$$P_j=\frac{b_0b_1\cdots b_{j-1}}{a_1a_2\cdots a_j(1+b_0e_1)}=\frac{\varepsilon^j}{j!}\frac{1}{1+be_1}=\frac{\varepsilon^j}{j!}\frac{1}{e^{\varepsilon}}(j>0) \quad (14)$$

If X denotes the droplets number in cell C, then the mean of X with stationary distribution is

$$E(X)=\sum_{j=0}^{\infty}jP_j=\varepsilon=\frac{b}{a} \quad (15)$$

IV. DISCUSSION

There are several points to be illuminated for the foregoing deduction and solution of the model equations.

1) In gas/drop two-phase flows, contains usually a certain amount of small droplets, and this makes the growth factor of the process is positive, that is $b > 0$, thus the foregoing discussion holds. If $b = 0$, that is, no droplet immigrating, a pure death process will generate.

2) For the dense gas/droplet flows, collision and coalescence of droplets are more complex, and usually cannot be described by a birth and death process.

3) From the mean of stationary distribution, it can be seen, the greater the number density of larger drops and the relative velocity between drops and droplets are, the smaller the mean value is. The greater the number density of small droplets upstream and their speeds are, the greater the mean value is.

4) Because the mean formula of stationary distribution contains the grid size Δx, it implies that a proper spatial scale, depending on the characteristic speed and length of flow field, to be selected to describe correctly the probability of moving droplets.

5) The created model in the paper is easy to generalize to two-dimensional case without any essential difficult.

V. CONCLUSION

Collision and coalescence of droplets is familiar and important phenomenon in gas/droplet two-phase flow, particularly in some flows that need to consider the mass exchange between two classes of drops with different sizes. In actual gas/droplet two-phase flows, containing many such classes, collision and coalescence of droplets enhance the complex of flows, thus, it is difficult to establish a general model to describe the changes of droplets number. Whereas, for only the two classes of bigger and smaller drops, the method of birth and death process proposed in this paper can be introduced to establish the model and to deduce a corresponding equation. A mean formula of droplets number obtained in this paper is convenient for application.

REFERENCES

[1] Shih-I Pai, Two-phase flows, Vieweg, 1977.

[2] Feller W., Introduction to probability theory and its applications, vol(1),1957.

[3] Satty,T.L., Elementary of queuing theory with applications,1961.

[4] Wang Z.Q.,Stochastic process theory, Beijing: Science Press, 1965. (in Chinese)