A Transfer Matrix Method for Horizontal Seismic Analysis of Tube-in-Tube Structure

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***Abstract*-In order to provide a new approximated method for the preliminary design, tub-in-tube structure is simplified as a double cantilever beam system with compatible horizontal displacement at each floor along one of orthogonal direction of structure. According to the linear elastic assumption, the fluctuation equations of tubes, the equilibrium of floor and displacement compatibility conditions, the spectrum of dynamic response of structure such as displacement, rotation angle, bending moment and shear force of two tubes are expressed by a transfer vector with 6 dimensionless elements. The equations to solve seismic response, natural frequency and mode shape of structure are finally derived by taking into account the boundary condition of structure. This method has 8 parameters in each storey and is suitable for structure with different storey parameters. With the aid of Fast Fourier Transform, it can provide time history solution for displacement, rotation angle, bending moment and shear force of every tube at the top and bottom of each storey. During the preliminary design, the designer can use this method to realize a satisfied design efficiently.**

Keywords-tube-in-tube; transfer matrix method; seismic response analysis; high-rise building; structure preliminary design

I. INTRODUCTION

Tube-in-tube structure commonly consists of an inner tube for vertical transportation demand and an outer tube with dense columns and deep beams. It is a very commonly used structural system for high-rise building more than 50 storeys. In order to improve the computational efficiency in the preliminary design, a great number of approximate analysis approaches [1~4] have been developed to substitute the finite element method which is elaborate and reliable but too exhausted in calculating effort and time. Most of the approximate model for horizontal vibration analysis takes the tube-in-tube structure as a double cantilever beam system with compatible deformation between the two tubes. On the basis of continuum parameter ideology, closed solution of the double beam system is obtained, especially when structural parameters are supposed to be constant along the height of structure. In order to provide a more adaptable simplified approach for the possible situation of varied storey parameters in preliminary design, this research make an effort on the transfer matrix method for horizontal vibration of tube-in-tube structure.

Transfer Matrix Method (TMM) has been widely used for seismic dynamic analysis of high-rise buildings with various structural system, such as frame structures [5], shear wall structures [6] and frame shear wall structures [7~9]. However, according to the author's scope, there hasn't any attempt of TMM on tube-in-tube structure. This paper proposes a simplified model of TMM and corresponding numerical procedure based on the double cantilever beam system, which depends on brief fundamental parameters of structure and can provide horizontal free vibration characteristics and elastic seismic response for tube-in-tube structure.

II. NUMERICAL MODEL AND PARAMETERS

A. *Assumptions and Numerical Analysis Model*

Consider the elastic vibration of tube-in-tube structure, which is appropriate for the analysis of free vibration characteristics and the performance investigation under frequent earthquake. Suppose that the floor of each storey is enough rigid within its plane and the structure has an orthogonal and symmetrical structure plane. Thus, when the structure vibrates horizontally along one of its orthogonal direction, the tube-in-tube structure can be taken as a plane system which consists of two bending members representing the two tubes respectively and horizontal rigid rods at each floor level, which means that the horizontal displacements of the two tube at each floor level are compatible, while other displacements of them at each floor level are independent.

B. *Structure Parameters*

Consider a tube-in-tube structure with N storeys. There are 8 parameters at storey i (i =1,2,…N): h_i (story height), G_i (mass concentrated at the floor level), ρ_{Wi} and ρ_{Fi} (mass density of inner tube and outer tube, where subscript "W" and "F" denote the inner one and outer one respectively), $E_{Wi}I_{Wi}$, $G_{Wi}A_{Wi}$, $E_{Fi}I_{Fi}$ and $G_{Fi}A_{Fi}$ (bending stiffness and shear stiffness of two tubes).

The inner tube is often made of reinforce concrete or steel reinforce concrete. Its bending stiffness $E_{Wi}I_{Wi}$ and shear stiffness $G_{Wi}A_{Wi}$ can be easily obtained according to the theory of thin-walled open section members. The outer framed tube, however, is a complicated space system and the approximate formulae of $E_{Fi}I_{Fi}$ and $G_{Fi}A_{Fi}$ can be found in those references which aim to explore the horizontal elastic characteristics of framed tube structure especially [10]. It should be mentioned that taking the outer framed tube as a bending member with only two representative parameters describing the horizontal vibration stiffness is the most important simplified issue of the model, which maybe give rise to the primary error resource of solution.

III. DERIVATION OF TRANSFER MATRIX AT FREQUENT DOMAIN

A. Relationship between Transfer Vector and Seismic Response

Taking each tube as bending member and using fluctuation equations of a bending member of story i in the frequent domain, seismic responses of two tubes at the bottom and top of story i can be expressed by the following equations:

$$\xi_{Fi}^{I} = A_{Fi}\zeta_{Fi}\,,\quad \xi_{Fi}^{J} = B_{Fi}\zeta_{Fi}\,,\quad \zeta_{Fi} = \begin{bmatrix} D_{1i} & D_{2i} & D_{3i} & D_{4i} \end{bmatrix}^{T}, \tag{1a,b,c}$$

$$\xi_{Fi}^{I} = \begin{bmatrix} U_{Fi}^{I} & \theta_{Fi}^{I} & M_{Fi}^{I} & V_{Fi}^{I} \end{bmatrix}^{T},\quad \xi_{Fi}^{J} = \begin{bmatrix} U_{Fi}^{J} & \theta_{Fi}^{J} & M_{Fi}^{J} & V_{Fi}^{J} \end{bmatrix}^{T}, \tag{1d,e}$$

$$\xi_{Wi}^{I} = A_{Wi}\zeta_{Wi}\,,\quad \xi_{Wi}^{J} = B_{Wi}\zeta_{Wi}\,,\quad \zeta_{Wi} = \begin{bmatrix} C_{1i} & C_{2i} & C_{3i} & C_{4i} \end{bmatrix}^{T}, \tag{2a,b,c}$$

$$\xi_{Wi}^{I} = \begin{bmatrix} U_{Wi}^{I} & \theta_{Wi}^{I} & M_{Wi}^{I} & V_{Wi}^{I} \end{bmatrix}^{T},\quad \xi_{Wi}^{J} = \begin{bmatrix} U_{Wi}^{J} & \theta_{Wi}^{J} & M_{Wi}^{J} & V_{Wi}^{J} \end{bmatrix}^{T}, \tag{2d,e}$$

where superscript "I" and "J" denote the bottom and top of storey respectively and U , θ , M and V denote horizontal displacement, rotation angle, moment and shear force spectrum. ζ_{Fi} and ζ_{Wi} are called the transfer vector, which has totally 8 status quantities (i.e., the 8 elements of them) without physics unit. The elements of A_i and B_i (whatever with subscript "W" or "F", the expressions are same) are expressed by E_iI_i , G_iA_i , ρ_i , h_i and vibration circular frequency ω :

$$[A_i]=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{\lambda_{1i}}{h_i}\left(1+\frac{\alpha_i}{\beta_i^4}\lambda_{2i}^2\right) & 0 & \frac{\lambda_{2i}}{h_i}\left(1-\frac{\alpha_i}{\beta_i^4}\lambda_{1i}^2\right) \\ E_iI_i\left(1+\frac{\alpha_i}{\beta_i^4}\lambda_{2i}^2\right)\frac{\lambda_{1i}^2}{h_i^2} & 0 & -E_iI_i\left(1-\frac{\alpha_i}{\beta_i^4}\lambda_{1i}^2\right)\frac{\lambda_{2i}^2}{h_i^2} & 0 \\ 0 & -\frac{E_iI_i}{h_i^3}\lambda_{1i}\lambda_{2i}^2 & 0 & \frac{E_iI_i}{h_i^3}\lambda_{2i}\lambda_{1i}^2 \end{bmatrix} \tag{3}$$

$$[B_i]=\begin{bmatrix} \cosh\lambda_{1i} & \sinh\lambda_{1i} & \cos\lambda_{2i} & \sin\lambda_{2i} \\ \frac{\lambda_{1i}}{h_i}\left(1+\frac{\alpha_i}{\beta_i^4}\lambda_{2i}^2\right)\sinh\lambda_{1i} & \frac{\lambda_{1i}}{h_i}\left(1+\frac{\alpha_i}{\beta_i^4}\lambda_{2i}^2\right)\cosh\lambda_{1i} & -\frac{\lambda_{2i}}{h_i}\left(1-\frac{\alpha_i}{\beta_i^4}\lambda_{1i}^2\right)\sin\lambda_{2i} & \frac{\lambda_{2i}}{h_i}\left(1-\frac{\alpha_i}{\beta_i^4}\lambda_{1i}^2\right)\cos\lambda_{2i} \\ E_iI_i\left(1+\frac{\alpha_i}{\beta_i^4}\lambda_{2i}^2\right)\frac{\lambda_{1n}^2}{h_i^2}\cosh\lambda_{1i} & E_iI_i\left(1+\frac{\alpha_i}{\beta_i^4}\lambda_{2i}^2\right)\frac{\lambda_{1i}^2}{h_i^2}\sinh\lambda_{1i} & -E_iI_i\left(1-\frac{\alpha_i}{\beta_i^4}\lambda_{1i}^2\right)\frac{\lambda_{2i}^2}{h_i^2}\cos\lambda_{2i} & -E_iI_i\left(1-\frac{\alpha_i}{\beta_i^4}\lambda_{1i}^2\right)\frac{\lambda_{2i}^2}{h_i^2}\sin\lambda_{2i} \\ -\frac{E_iI_i}{h_i^3}\lambda_{1i}\lambda_{2i}^2\sinh\lambda_{1i} & -\frac{E_iI_i}{h_i^3}\lambda_{1i}\lambda_{2i}^2\cosh\lambda_{1i} & -\frac{E_iI_i}{h_i^3}\lambda_{2i}\lambda_{1i}^2\sin\lambda_{2i} & \frac{E_iI_i}{h_i^3}\lambda_{2i}\lambda_{1i}^2\cos\lambda_{2i} \end{bmatrix} \tag{4}$$

where

$$\alpha_i = \frac{\rho_i\omega^2}{G_iA_i}h_i^2\,,\quad \beta_i^4 = \frac{\rho_i\omega^2}{E_iI_i}h_i^4\,,\quad \lambda_{1i} = \sqrt{-\frac{\alpha_i}{2}+\sqrt{\frac{\alpha_i^2}{4}+\beta_i^4}}\,,$$

$$\lambda_{2i} = \sqrt{\frac{\alpha_i}{2}+\sqrt{\frac{\alpha_i^2}{4}+\beta_i^4}}\,. \tag{5}$$

B. Equilibrium and Displacement Compatibility Conditions

At the top and bottom of storey i and $i+1$, the displacement compatibility between two tubs requires that

$$U_F^I(i)=U_W^I(i)\,,\quad U_F^J(i)=U_W^J(i)\,, \tag{6}$$

$$U_F^I(i+1)=U_W^I(i+1)\,,\quad U_F^J(i+1)=U_W^J(i+1)\,. \tag{7}$$

The continuous requirements of displacements of each tube need that

$$U_F^I(i+1)=U_F^J(i) \text{ or } U_W^I(i+1)=U_W^J(i) \text{ (one of them)}, \tag{8}$$

$$\Theta_F^I(i+1)=\Theta_F^J(i)\,,\quad \Theta_W^I(i+1)=\Theta_W^J(i)\,. \tag{9}$$

The equilibrium conditions of floor i with concentrated mass G_i are

$$M_F^I(i+1)=M_F^J(i)\,,\quad M_W^I(i+1)=M_W^J(i)\,, \tag{10}$$

$$V_F^I(i+1)+V_W^I(i+1)=V_F^J(i)+V_W^J(i)-G(i)\omega^2U_W^J(i)\,. \tag{11}$$

where the moment of inertia of floor are neglected.

C. Transfer Matrix of Structure

It can be seen that there are 16 status quantities at storey i and i +1. Eq.6~11 provide 10 related expressions between them. Thus, one can arbitrarily choose 6 basic elements to express other 10 elements. Here is the result where

$$\zeta_i = \begin{bmatrix} C_{1i} & C_{2i} & C_{3i} & C_{4i} & D_{2i} & D_{4i} \end{bmatrix}^T \tag{12}$$

is adopted as the basic transfer vector:

$$\zeta_{i+1} = t_{i+1,i}\zeta_i\,,\quad \begin{bmatrix} D_{1i} & D_{3i} \end{bmatrix}^T = z_i\zeta_i\,, \tag{13}$$

where $t_{i+1,i}$ is call the transfer matrix between adjacent storey. The obtained expressions of element of $t_{i+1,i}$ and z_i contains a large amount of constants expressed by the 16 parameters of storey i and i +1, which are not listed here due to the limited space. One can refer to [11]. The transfer relationship between ζ_i and ζ_1 can be easily deduced by using Eq.10 from storey 1 to i successively:

$$\zeta_i = T_i\zeta_1\,,\quad T_i = t_{i,i-1}t_{i-1,i-2}\ldots t_{2,1} \quad (i = 2,3,\ldots N), \tag{14}$$

where T_i is called as transfer matrix of structure.

IV. SEISMIC RESPONSE SOLUTION

Suppose that the bottoms of two tubes are fixed end which move with the free ground motion when horizontal earthquake occurs. The boundary conditions at the structure bottom are:

$$U_{W1}^I = U_f\,,\quad \Theta_{F1}^I = 0\,,\quad \Theta_{W1}^I = 0\,, \tag{15}$$

where U_f is the displacement spectrum of seismic ground motion. The boundary conditions at the top of structure are

$$0=V_F^J(N)+V_W^J(N)-G(N)\omega^2U_W^J(N)\,,\quad M_F^J(N)=0\,,$$

$$M_W^J(N)=0\,. \tag{16}$$

Substituting Eq.3a into Eq.15, one can obtain 3 relationships between $\zeta_i(1)$ (denote as Group-1 expressions). By using Eq.3b and Eq.14 and letting i =N, Eq.16 can be written as 3 relationships between $\zeta_i(1)$ (denote as Group-2 expressions). The combination of 6 expressions of Group-1 and Group-2 give rise to the expression of elements of $\zeta_i(1)$ as following:

$$\begin{bmatrix} \bar{A} & \bar{B} & \bar{F} \\ \hat{A} & \hat{B} & \hat{F} \\ \breve{A} & \breve{B} & \breve{F} \end{bmatrix} \begin{Bmatrix} C_1(1) \\ C_2(1) \\ D_2(1) \end{Bmatrix} = \begin{Bmatrix} -\bar{x}_N \\ -\hat{x}_N \\ -\breve{x}_N \end{Bmatrix} U_f\,, \qquad (17)$$

$$C_3(1) = U_f - C_1(1),\ D_4(1) = -\frac{\tilde{c}_{F1}}{\tilde{d}_{F1}} D_2(1),\ C_4(1) = -\frac{\tilde{c}_{W1}}{\tilde{d}_{W1}} C_2(1) \qquad (18)$$

As soon as $\zeta_i(1)$ is solved by Eq.17 and Eq.18, all of the seismic response of each storey in the frequent domain listed in Eq.1 and Eq.2 can be calculated by Eq.14 and Eq.13 successively. With the aid of Fast Fourier Transform (FFT) technique, one can obtain the time history of seismic response easily. The constants in Eq.17 and Eq.18 can be found in [11].

V. FREQUENCY AND MODE SHAPE

When the structure is at free vibration stage, U_f =0. To ensure Eq.17 having non-zero solution, following expression must exist:

$$\Delta = \bar{A}\hat{B}\breve{F} + \hat{A}\breve{B}\bar{F} + \breve{A}\bar{B}\hat{F} - \breve{A}\hat{B}\bar{F} - \bar{A}\breve{B}\hat{F} - \hat{A}\bar{B}\breve{F} = 0\,. \quad (19)$$

As all the expressions of parameter in Eq.19 contain unknown ω and other known constants expressed by 8 parameters of every storey, Eq.19 is the natural frequency equation. It is a high order algebraic equation about ω. Scanning-check method is recommended with a trial range of ω, where the frequency solutions can be determined by finding the intersection points between $\omega - \Delta$ curve and ω axis. After a definite frequency ω is obtained, let U_f =0 and $C_1(1) = 1$ in Eq.17 and Eq.18, one can obtain that (all the constants seen in [11]):

$$C_1(1) = 1,\ C_2(1) = \frac{\hat{F}\breve{A} - \breve{F}\hat{A}}{\hat{B}\breve{F} - \breve{B}\hat{F}} = \frac{\dfrac{\breve{A}}{\breve{F}} - \dfrac{\hat{A}}{\hat{F}}}{\dfrac{\hat{B}}{\hat{F}} - \dfrac{\breve{B}}{\breve{F}}},\ C_3(1) = -1, \quad (20a,b,c)$$

$$C_4(1) = -\frac{\tilde{c}_{W1}}{\tilde{d}_{W1}} C_2(1),\ D_2(1) = \frac{\breve{B}\hat{A} - \hat{B}\breve{A}}{\hat{B}\breve{F} - \breve{B}\hat{F}} = \frac{\dfrac{\hat{A}}{\hat{B}} - \dfrac{\breve{A}}{\breve{B}}}{\dfrac{\breve{F}}{\breve{B}} - \dfrac{\hat{F}}{\hat{B}}},$$

$$D_4(1) = -\frac{\tilde{c}_{F1}}{\tilde{d}_{F1}} D_2(1) \quad (20d,e,f)$$

$$C_1(1) = 1,\ C_2(1) = -\bar{A}/\bar{B},\ C_3(1) = -1,\ C_4(1) = \tilde{c}\bar{A}_1/(\tilde{d}_1\bar{B}) \quad (20g,h,i,j)$$

Substituting Eq.20 into Eq.14 and then into Eq.13, the displacement at each storey can be calculated, which illustrate the mode shape of vibration of structure.

VI. EXAMPLE

To illustrate the propose method, Example 1 in reference [1] was calculated after a general computer program written in Matlab was completed. The example structure is a 50 storey reinforced concrete structure with a framed inner tube and a framed outer tube. The outer tube dimension is 50m×30m and the inner one is 20m×10m, where the short side is parallel to the direction of vibration. The center to center spacing of columns is 2.5m, and the total number of column of inner tube and outer tube are 24 and 64, respectively. Other structural parameters are: storey height 3.0m, beam and column section 0.8m×0.8m, floor thickness 0.25m, elastic modulus E =2×107kPa, Poisson ratio 0.25, material density 25kN/m3. 8 of the TMM parameters of each storey are listed in Table1.

The first two frequencies results are shown in Table 2. The error between the results of SAP2000 and TMM can be explained as the approximate estimate of $E_{Wi}I_{Wi}$, $G_{Wi}A_{Wi}$, $E_{Fi}I_{Fi}$ and $G_{Fi}A_{Fi}$ due to neglecting the effect of shear lag [1]. Compared to the results of the model proposed in [1], the distributed mass within the storey height and the concentrated mass at floor level are determined strictly in TMM, while in [1], their summation is distributed factitiously along the height of structure because a homogeneous and continuum structure is necessary for the model. Fig.1 illustrates some solutions under EL_Centro earthquake wave, where the maximum ground acceleration is 70gal.

TABLE I .PARAMETERS OF THE TRANSFER MATRIX METHOD.

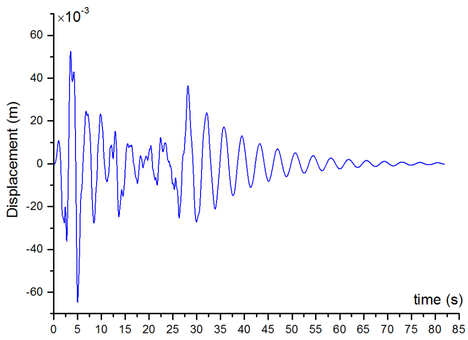
Parameter	**Value**	**Notes**
h_i	3.0 m	/
G_i	1315.816 t	[0.25×(30×50)×25 +(24+64) ×(0.8×0.8)×2.5×25]/9.8
ρ_{Fi}	104.490 t/m	(0.8×0.8)×25×64/9.8
ρ_{Wi}	39.184 t/m	(0.8×0.8)×25×24/9.8
$E_{Fi}I_{Fi}$	1.3512×10^{11} kN m^2	Given directly in the context of [1]
$G_{Fi}A_{Fi}$	2.9852×10^{7} kN	Given directly in the context of [1]
$E_{Wi}I_{Wi}$	5.5912×10^{9} kN m^2	Given directly in the context of [1]
$G_{Wi}A_{Wi}$	1.1482×10^{7} kN	Given directly in the context of [1]

TABLE II . FIRST TWO NATURAL FREQUENCIES (RAD/S).

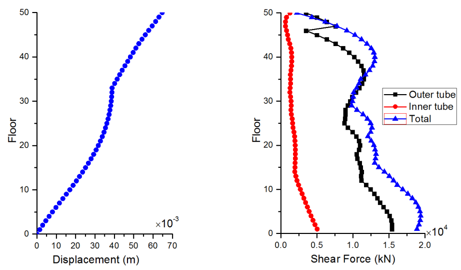
Type of Model	**SAP2000 [1]**		**model in [1]**		**TMM (this paper)**	
	ω_1	ω_2	ω_1	ω_2	ω_1	ω_2
Value	1.239	5.838	1.398	5.88	1.757	6.184

VII. CONCLUSION

This paper presents a transfer matrix method to solve frequency and mode shape of horizontal free vibration and horizontal seismic response of tube-in-tube structure. The assumption of linear elastic deformation is the prerequisite because FFT technique is used for the calculation of dynamic response, which means the application of superposition principle.



(A) TIME HISTORY OF HORIZONTAL DISPLACEMENT AT THE TOP OF STRUCTURE



(B) MAXIMUM DISPLACEMENT AND SHEAR FORCE AT FLOOR LEVEL

FIGURE I. SEISMIC RESPONSE SOLUTIONS.

The proposed simplified model of TMM has 8 parameters in each storey and is suitable for the case of different flexural rigidity, mass and storey height. It can provide time history for displacement, rotation angle, bending moment and shear force of every tube at the top and bottom of each storey. During the preliminary design, the designer can use this method, check the structure dynamic response under specific earthquake and adjust the parameters to realize a satisfied design, which is ofter controlled by the maximum deformation of structure.

To use this approach, a computer programme should be made and the scanning-check method to determine frequency is recommended. Detailed expressions of those constants in the equations of this paper need to refer to [14]. The reliability of the numerical results primarily depends on the accurate approximation of $E_{Wi}I_{Wi}$, $G_{Wi}A_{Wi}$, $E_{Fi}I_{Fi}$ and $G_{Fi}A_{Fi}$. Although the reliability of the approach need to be investigated by more examples, the real description way for distributed mass within storey height and concentrated mass at floor level is more reasonable than those continuum methods based on the closed solution of differential equations of structure.

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