

The Pricing for Warrant Bonds under Fractional Brownian Motion

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Abstract—Assuming that the underlying stock follows Fractional Brownian motion and that stochastic interest rate meets the Vasicek model of interest rates, this paper establishes pricing model of Warrant Bonds and deduces the pricing formula of Warrant Bonds by utilizing risk-neutral valuation theory. Finally, this paper analyzes influence of concerned parameters of pricing model on the value of Warrant Bonds by using the numerical simulations.

Keywords—warrant bonds; fractional brownian motion; option; risk-neutral pricing theory

I. INTRODUCTION

Warrant Bonds (referred WBS) is one kind of special convertible bonds which is traded separately in equity warrants and bonds, and which is usually issued by listed company. WBS consists of two parts: bond and the stock warrant. WBS gives the holders the right to buy issuing company's stocks in a certain period of time prior to expiration for the exercise price and exercise ratio provided by the issuing company under the premise of maintaining the validity of the bonds when it is issued. There is essential difference between WBS and ordinary convertible bonds(CBS) which lies in separable transactions about bond and option. After the holders of the WBS exercise the warrants rights, their claims still exist, and they can still recover the principal by the expiration date and earn the interest. However, the creditors' rights of the investors of CBS do not exist after their rights of warrant are exercised. Thus, WBS can be understood as a kind of innovative financial product of "obtaining warrants while buying bonds".

When WBS first appeared in the United States in 1970s, many foreign scholars believed that it would become an alternative of the ordinary convertible bonds. Ingersoll (1977)[1] proposed that bonds and warrants be combined with a financing portfolio equivalent to convertible bonds. Finnerty (1986)[2] conducted a questionnaire for institutional investors buying WBS. The results showed that the discount of initial issuance price of WBS would bring about the tax burden to the investors. And Payne [3] made a comprehensively comparative study of correlation between WBS and CBS. In recent years, Chinese scholars have also made some positive exploration on the pricing of WBS. K. Xu and T. Li (2007) [4] made an empirical analysis of the pricing for China's first WBS based on Ma an shan Iron convertible bonds. H.Y. Hua, X.J. Cheng (2007) [5] considered that the warrant in WBS is a simple Bermuda warrant. They gave a theoretical price of

warrants by using martingale pricing theory under a complete market. Z.H. Li(2008) [6] made a study of the pricing for WBS of Bao steel issued in 2008 by using B-S model of the dilution effect, and concluded that the pricing of the bond part of WBS of Bao steel performed well, but the price of the warrant is high. H. Luo, H.M. Shen [7] studied the practical application of WBS in our country. D. Zhu (2011)[8] used martingale pricing methods to derive the pricing formula of WBS. But there is an inadequacy, that is assuming that the underlying stock price follows Geometric Brownian motion, then it means that the changes of stock price are independent random variables, and that yield rate on assets follows normal distribution, which does not accord with the actual fluctuation of stock price. Because a large number of empirical studies and behavioral finance researches have shown that the fluctuation of the stock price is not random walk, but exhibits varying degrees of long-range dependence in different time; and the distribution of asset returns is not normal, but is characterized by "high kurtosis and fat tail"; this is a typical feature of financial asset returns.

This paper will improve the pricing model of WBS established by D. Zhu, and derive the pricing formula of WBS by using risk-neutral valuation theory. Since Peter (1994) has proved the assumption that Fractional Brownian motion(FBM) does not depend on independence and normal distribution, its characteristic index and scale parameters can describe well the volatility of the financial markets, stock prices process and the characteristics of "high kurtosis and fat tail" of asset yields' distribution and so on. In addition, its self-similarity and long-range dependence is consistent with people's intuition of the financial markets, that is to say, future stock price will depend on not only the current price of the stock, but also the price over a considerable period of time [9]. Therefore, in order to more objectively reflect the reality of financial markets, this paper utilizes FBM to capture the fluctuations of the underlying stock price.

II. PRELIMINARIES

The value of WBS consists of two parts: pure bond value and value of call option of the underlying stock. There is formula of maturity cash flow as follows:

$$V_T = \begin{cases} P_b & , S_T \leq \frac{P_b \cdot C_u}{M} \\ P_b + \alpha \beta (S_T - C_u) & , S_T > \frac{P_b \cdot C_u}{M} \end{cases}$$

Where $P_b = Me^{iT}$ denotes pure bond value calculated through coupon rate i (constant), M denotes the face value

of convertible bonds, C_v denotes the agreed exercise price (the conversion price), T denotes time to expiration of Warrant Bonds, S_T denotes the stock price at time T , α denotes the ratio of the warrants attached to bonds (the amount of warrant which bonds subscribers can be free to obtain while buying one bond), and β denotes the right proportion (the number of underlying stock which one warrant assures the holders can purchase).

III. FINANCIAL MARKET MODEL

A. The Pricing Model for Warrant Bonds

(1) Let risk-free interest rate $r(t)$ be random, and meet the Vasicek model commonly used in finance,

$$dr(t) = k(\theta - r(t))dt + \delta dB(t) \quad (1)$$

Where k means reversion rate, θ denotes long-term interest mean, k, θ, δ are positive constants, $B(t)$ is a standard Brownian motion.

(2) The basic stock price corresponding to Warrant Bonds follows Fractional Brownian motion process, and satisfies the following differential equation

$$dS(t) = \mu(t)S(t)dt + \sigma S(t)dB_H(t), \quad 0 \leq t \leq T \quad (2)$$

Where $\mu(t)$ denotes the expected rate of return on stock price at time t , σ denotes the stock price volatility, $\mu(t)$ is a function of t ; $B_H = \{B_H(t, \omega), t > 0\}$ is a FBM whose Hurst parameter is H , T denotes time to expiration, assuming $B(t)$ and $B_H(t)$ are mutually independent. In the risk-neutral world, we obtain $\mu(t) = r(t)$, then equation (2) can now be written as follows:

$$dS(t) = r(t)S(t)dt + \sigma S(t)dB_H(t), \quad 0 \leq t \leq T \quad (3)$$

The following differential equation can be obtained by using the Itô formula:

$$d \ln S(t) = (r(t) - \sigma^2 H t^{2H-1})dt + \sigma dB_H(t)$$

Accordingly, for all $0 \leq t \leq T$,

$$S(T) = S(t) \exp\left\{\int_t^T r(s)ds - \frac{1}{2}\sigma^2(T^{2H} - t^{2H}) + \sigma(B_H(T) - B_H(t))\right\} \quad (4)$$

B. Lemma 1

[8] Solution of stochastic differential equation (1) is

$$r(u) = (r(t) - \theta)e^{-k(u-t)} + \theta + \delta \int_t^u e^{-k(u-s)} dB(s), \quad \text{for } t \leq u \leq T$$

$$\text{and } \int_t^T r(u)du = \frac{r(t)-\theta}{k}(1-e^{-k(T-t)}) + \theta(T-t) + \int_t^T \frac{\delta}{k}(1-e^{-k(T-s)})dB(s)$$

Proof refers to literature [8]

Let

$$A(t, T) = \frac{r(t)-\theta}{k}(1-e^{-k(T-t)}) + \theta(T-t), \quad Z(t, T) = \int_t^T \frac{\delta}{k}(1-e^{-k(T-s)})dB(s)$$

$$\text{Hence, } \int_t^T r(u)du = A(t, T) + Z(t, T) \quad (5)$$

From the literature [10],

$$\text{var}[Z(t, T)] = E[Z^2(t, T)] = \int_t^T \left[\frac{\delta}{k}(1-e^{-k(T-s)})\right]^2 ds$$

$$= \frac{\delta^2}{2k^3}[-3 - e^{-2k(T-t)} + 4e^{-k(T-t)} + 2k(T-t)] A \sigma^2(t, T)$$

and $E[Z(t, T)] = 0$, Namely, $Z(t, T) : N(0, \sigma^2(t, T))$

IV. THE PRICING FORMULA OF WARRANT BONDS UNDER FRACTIONAL BROWNIAN MOTION

A. Lemma 2

[11] Provided $Z_1 : N(0, 1)$, $Z_2 : N(0, 1)$, $\text{Cov}(Z_1, Z_2) = \rho$, for any real number a, b, c, d, s , then the following equality is obtained:

$$E(e^{cZ_1 + dZ_2} I_{\{aZ_1 + bZ_2 \geq s\}}) = e^{\frac{1}{2}(c^2 + d^2 + 2\rho cd)} \Phi\left(\frac{ac + bd + \rho(ad + bc) - s}{\sqrt{a^2 + b^2 + 2\rho ab}}\right)$$

Where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}s^2} ds$ denotes the function of standard normal distribution, proof refers to literature [11].

B. Theorem

Assuming that random interest rate meets Vasicek rate model and the underlying stock price follows the Fractional Brownian motion process, the value of Warrant Bonds at any time t before expiration is:

$$V(t, T, r(t), S(t)) = P_b e^{-A(t, T) + \frac{1}{2}\sigma^2(t, T)} + \alpha\beta S(t) \Phi\left(\frac{\frac{1}{2}\sigma^2(T^{2H} - t^{2H}) - \ln \frac{P_b C_v}{M S_t} + A(t, T)}{\sqrt{\sigma^2(t, T) + \sigma^2(T^{2H} - t^{2H})}}\right) - \alpha\beta C_v e^{-A(t, T) + \frac{1}{2}\sigma^2(t, T)} \Phi\left(-\frac{\ln \frac{P_b C_v}{M S_t} - A(t, T) + \sigma^2(t, T) + \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sqrt{\sigma^2(t, T) + \sigma^2(T^{2H} - t^{2H})}}\right) \quad (6)$$

Proof. From the risk-neutral pricing theory

$$V(t, T, r(t), S(t)) = E[e^{-\int_t^T r(u)du} \cdot V_T | F_t] \\ = E[e^{-\int_t^T r(s)ds} P_b I_{\{S_T \leq \frac{P_b C_v}{M}\}} | F_t] + E[e^{-\int_t^T r(u)du} (P_b + \alpha\beta(S_T - C_v)) I_{\{S_T > \frac{P_b C_v}{M}\}} | F_t]$$

Let V_1 and V_2 respectively denote the first part and second part above formula.

First, calculate V_1 ,

$$V_1 = P_b E[e^{-\int_t^T r(s)ds} I_{\{S_T \leq \frac{P_b C_v}{M}\}} | F_t] = P_b e^{-A(t, T)} E[e^{-Z(t, T)} I_{\{S_T \leq \frac{P_b C_v}{M}\}} | F_t] \\ = P_b e^{-A(t, T)} E[e^{-Z(t, T)} I_{\{\exp[A(t, T) - \frac{1}{2}\sigma^2(T^{2H} - t^{2H}) + Z(t, T) + \sigma(B_H(T) - B_H(t))] \leq \frac{P_b C_v}{M S_t}\}} | F_t]$$

Notice that: $Z(t, T) : N(0, \sigma^2(t, T))$,

$$B_H(T) - B_H(t) : N(0, T^{2H} - t^{2H}) \quad [12]$$

And then $\frac{Z(t,T)}{\sigma(t,T)} : N(0,1)$, $\frac{B_H(T)-B_H(t)}{\sqrt{T^{2H}-t^{2H}}} : N(0,1)$

And since $B(t)$ and $B_H(t)$ are mutually independent, then $Cov\left(\frac{Z(t,T)}{\sigma(t,T)}, \frac{B_H(T)-B_H(t)}{\sqrt{T^{2H}-t^{2H}}}\right) = 0$.

Thus, from Lemma 2, get:

$$\begin{aligned} V_1 &= P_b e^{-A(t,T)} E[e^{-\frac{Z(t,T)}{\sigma(t,T)} I_{\left\{\frac{\sigma(t,T)Z(t,T)}{\sigma(t,T)} + \sigma\sqrt{T^{2H}-t^{2H}} \cdot \frac{B_H(T)-B_H(t)}{\sqrt{T^{2H}-t^{2H}}} \leq \ln \frac{P_b C_v}{M S_t} - A(t,T) + \frac{1}{2}\sigma^2(T^{2H}-t^{2H})\right\}} | F_t]} \\ &= P_b e^{-A(t,T) + \frac{1}{2}\sigma^2(t,T)} \Phi\left(\frac{\ln \frac{P_b C_v}{M S_t} - A(t,T) + \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + \sigma^2(t,T)}{\sqrt{\sigma^2(t,T) + \sigma^2(T^{2H}-t^{2H})}}\right) \end{aligned} \quad (7)$$

Then calculate V_2 ,

$$V_2 = (P_b - \alpha\beta C_v) E(e^{-\int_t^T r(u)du} I_{\{S_T \geq C_v\}} | F_t) + \alpha\beta E(e^{-\int_t^T r(u)du} S_T I_{\{S_T \geq C_v\}} | F_t) \quad (8)$$

Let V_{21} and V_{22} respectively denote the first part and second part of above formula (8), V_{21} is calculated by the same methods of calculation of V_1 , get:

$$V_{21} = (P_b - \alpha\beta C_v) e^{-A(t,T) + \frac{1}{2}\sigma^2(t,T)} \Phi\left(\frac{\ln \frac{P_b C_v}{M S_t} - A(t,T) + \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + \sigma^2(t,T)}{\sqrt{\sigma^2(t,T) + \sigma^2(T^{2H}-t^{2H})}}\right) \quad (9)$$

From formula (5), get:

$$\begin{aligned} V_{22} &= \alpha\beta E[e^{-\int_t^T r(u)du} S_T I_{\{S_T \geq \frac{P_b C_v}{M S_t}\}} | F_t] = \alpha\beta e^{-A(t,T)} E[e^{-Z(t,T)} S_T I_{\{S_T \geq \frac{P_b C_v}{M S_t}\}} | F_t] \\ &= \alpha\beta S(t) e^{-A(t,T) - \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + \sigma(B_H(T)-B_H(t))} \cdot I_{\left\{e^{A(t,T) - \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + Z(t,T) + \sigma(B_H(T)-B_H(t))} > \frac{P_b C_v}{M S_t}\right\}} | F_t \\ &= \alpha\beta S(t) e^{-\frac{1}{2}\sigma^2(T^{2H}-t^{2H})} E[e^{\sigma\sqrt{T^{2H}-t^{2H}} \cdot \frac{B_H(T)-B_H(t)}{\sqrt{T^{2H}-t^{2H}}}] | F_t] \\ &= \alpha\beta S(t) \Phi\left(\frac{\frac{1}{2}\sigma^2(T^{2H}-t^{2H}) - \ln \frac{P_b C_v}{M S_t} + A(t,T)}{\sqrt{\sigma^2(t,T) + \sigma^2(T^{2H}-t^{2H})}}\right) \end{aligned} \quad (10)$$

V_1 , V_{21} and V_{22} are respectively substituted in $V(t, T, r(t), S(t))$, get

$$\begin{aligned} V(t, T, r(t), S(t)) &= P_b e^{-A(t,T) + \frac{1}{2}\sigma^2(t,T)} + \alpha\beta S(t) \Phi\left(\frac{\frac{1}{2}\sigma^2(T^{2H}-t^{2H}) - \ln \frac{P_b C_v}{M S_t} + A(t,T)}{\sqrt{\sigma^2(t,T) + \sigma^2(T^{2H}-t^{2H})}}\right) \\ &\quad - \alpha\beta C_v e^{-A(t,T) + \frac{1}{2}\sigma^2(t,T)} \Phi\left(-\frac{\ln \frac{P_b C_v}{M S_t} - A(t,T) + \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + \sigma^2(t,T)}{\sqrt{\sigma^2(t,T) + \sigma^2(T^{2H}-t^{2H})}}\right) \end{aligned}$$

V. SIMULATION STUDIES

The aim of this section is to show how to implement FBM model for WBS and illustrates the effects of parameters of pricing model. For these purposes, let's make a report on two sets of numerical experiments. In the first set, we compare the theoretical prices calculated by the following models:

Geometric Brownian motion (hereafter GBM) and FBM. These tests will consist of some simulation of the above two pricing models with some chosen parameters. In the second set, we analyze the influences of different parameters in FBM model on the value of WBS. The following results and concerned figures are obtained by using Matlab.

A. Comparison of WBS Prices Calculated by GBM Model and FBM Model

Now, for an illustration of the differences between the two models: GBM and FBM, first, we compute the theoretical prices of WBS under the two models. Table 1 presents the parameters for computing the theoretical prices of WBS. Apart from H , the other corresponding parameters about GBM model and FBM model are the same. The prices computed by the above two pricing models are also presented in Table 2, where S_t denotes the underlying stock price at time t ; P_{GBM} denotes the prices of WBS computed by the GBM model; P_{FBM} denotes the prices of WBS calculated according to FBM model.

By comparing columns P_{GBM} and P_{FBM} in Table 2, the conclusions that we come to are (1) no matter it is about the low time or high time to maturity, the prices given by FBM model are larger than the prices given by GBM model. This is mainly because the value of stock warrant in WBS will increase when Hurst parameter H is greater than 0.5; (2) in the low time to maturity case, the prices calculated by FBM model and GBM model are close to each other, the differences are small. However, in high time to maturity case, the prices computed by FBM model are larger than the prices computed by GBM model. This is mainly because the value of stock warrant in WBS is the increasing function of the time to maturity, so the prices of WBS will increase; (3) apparently, with the increase of S_t , the prices obtained by GBM and FBM will both increase too, but the increasing amount is very small. This is because the value of the bonds which is the main component of WBS is very large and the value of the stock warrant is very small when compared with other derivatives.

TABLE I. THE VALUATION OF THE CHOSEN PARAMETERS USED IN THESE MODELS.

Model type	k	θ	r	δ	i	α	β	H	M	C_v
GBM	0.8	0.05	0.06	0.2	0.05	0.4	0.5	0.5	100	25
FBM	0.8	0.05	0.06	0.2	0.05	0.4	0.5	0.7	100	25

TABLE II. PRICING RESULTS OF DIFFERENT PRICING MODELS.

Underlying stock price: S_t	Low time to maturity: $T=2, t=1$		High time to maturity: $T=4, t=1$	
	P_{GBM}	P_{FBM}	P_{GBM}	P_{FBM}
15	105.0691	105.3500	109.4193	109.8588
20	105.7199	106.0474	110.1828	110.6666
25	106.4081	106.7623	110.9542	111.4762
30	107.1004	107.4789	111.7266	112.2861
35	107.7941	108.1962	112.4989	113.0959
40	108.4952	108.9168	113.2716	113.9059
45	109.2090	109.6437	114.0453	114.7161
50	109.9388	110.3788	114.8208	115.5267

B. The Influence of Parameters in FBM Model

In what follows, we will illustrate the influence of different parameters of FBM model on the value of WBS. For the sake of simplicity, we will just consider the in-the-money case. Indeed, using the same method, one can also discuss the remaining cases: out-of-the-money and at-the-money. We take a first look at the influence of different Hurst parameter H on the values of WBS and then consider the influence of other parameters σ, C_v and T on the values of WBS. The concerned parameters are $\theta=0.05, k=0.8, \delta=0.2, T=2, t=1, \sigma=0.8$,

$r=r(1)=6\%, H=0.7, i=5\%, \alpha=0.4, \beta=0.5, M=100, C_v=25$ 和 $S_t=26$, It only takes less than 10s to generate all the pictures in Figure.1 and Fig 2 on a high-performance computer. Not surprisingly, Figure.1 indicates :(1) the value of WBS is an increasing function of H, σ and T ; and (2)the increasing parameter of C_v comes along with a decrease of the value of WBS.

Figure.2 clearly displays the Value of WBS under different times to maturity and exercise prices, namely, a three dimensional functional image of the value of WBS with regard to time to maturity and exercise price. Figure.2 indicates that the value of WBS will increase with the increase of time to maturity and decrease with the increase of the exercise price.

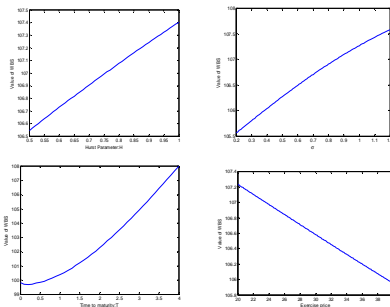


FIGURE I. VALUE OF WBS.

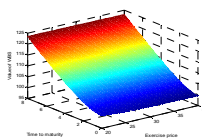


FIGURE II. VALUE OF WBS UNDER DIFFERENT TIMES TO MATURITY AND EXERCISE PRICES.

VI. CONCLUSION

This paper has overcome the drawback of the pricing model for WBS proposed by some predecessors, used the experiences for reference that the financial assets price follows Fractional Brownian motion process which a large number of financial empirical studies have proved, and established the pricing model for WBS which draws financial markets closer to the actual conditions of financial markets. Assuming that the underlying stock price follows Fractional Brownian motion and stochastic interest rate meets Vasicek rate model, the pricing formula of Warrant Bonds is obtained by utilizing the risk-neutral pricing theory (namely, no arbitrage pricing

theory). For young Chinese securities market, WBS is still a new financial derivative product which has great development potential. It will play a positive role in broadening financing channels for enterprises, enriching variety of securities and booming the securities market. Therefore, how to price scientifically for WBS will be very important, which is worthwhile for us to explore and study together.

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