Variations of Bergman Kernels for Some Explicitly Given Families of Planar Domains

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Abstract——**We study the parameter dependence of the Bergman kernels on some planar domains depending on complex parameter ζ in nontrivial ``pseudoconvex'' ways. Smoothly bounded cases are studied at first: It turns out that, in an example where the domains are discs, the Levi form for the logarithm of the Bergman kernels with respect to ζ approaches to ∞ as the point tends to the complement of a point in the boundary. Further, in contrast to this, in the cases where the boundaries of the domains are nonsmooth, such as discs with slits interesting phenomena isobserved.**

Keywords-bergman kernel; plurisubharmonic function; pseudoconvex domain

I. INTRODUCTION AND RESULTS

Bergman kernel and Bergman metric have been studied in detail in the case of bounded strongly pseudoconvex domains with C^∞ boundary. For such domains C. Fefferman [4] (1974) found a remarkable asymptotic formula for the Bergman kernel form. He used it to show that suitable geodesics of the Bergman metric approach the boundary of the domain in a ``pseudotransverse'' manner, and therefore that biholomorphicmappings of strongly pseudoconvex domains extend smoothly to the boundaries. In 1978, using Fefferman's asymptotic formula, P.Klembeck[7] showed, for a bounded strongly pseudoconvex domain Ω with C^∞ smooth boundary, that the holomorphic sectional curvature of Bergman pseudometric near the boundary of Ω approaches to the negative constant $-2/(n+1)$ which is the holomorphic sectional curvature of the Bergman metric of the unit ball in C^n. For some other results on the boundary behavior of the Bergman kernel, refer to [3], [5], [6], [9].

In 2004, F. Maitani and H. Yamaguchi [8] brought a new viewpoint by studying the variation of the Bergman metrics on the Riemann surfaces. Let us briefly recall it.

Let B be a disk in the complex ζ -plane, D be a domain in the products pace $B \times \mathbb{C}_z$, and let π be the first projection from $B \times \mathbb{C}_z$ to B which is proper and smooth and $D_\zeta = \pi^{-1}(\zeta)$ be a domain in $\mathbb{C}_z$. Put

$$\partial D = \bigcup_{\zeta \in B}(\zeta, \partial D_\zeta).$$

Let $\zeta \in B$, $z \in D_\zeta$ and consider the potential $\psi(\zeta, t, z)$ for (D_ζ, z) which is a complex valued harmonic function on $D_\zeta \setminus \{z\}$ vanishing on the boundary of D_ζ, and decompose $d_t\psi(\zeta, t, z)$ into

$$d_t\psi(\zeta, t, z) = \mathrm{L}(\zeta, t, z)dt + \overline{\mathrm{K}(\zeta, t, z)dt}$$

On $D_\zeta \setminus \{z\}$, where

$$\mathrm{L}(\zeta, t, z) = \frac{\partial \psi(\zeta, t, z)}{\partial t}, \mathrm{K}(\zeta, t, z) = \frac{\overline{\partial \psi(\zeta, t, z)}}{\partial t}.$$

Let $g(\zeta, t, z)$ be the Green function and $\lambda(\zeta, z)$ the Robin constant for (D_ζ, z), then,

$$g(\zeta, t, z) = \log\frac{1}{|t-z|} + \lambda(\zeta, z) + h(\zeta, t, z).$$

Here, $h(\zeta, t, z)$ is harmonic for t in a neighborhood of z in D_ζ such that $h(\zeta, z, z) = 0$ for $\zeta \in B$.

Let $\varphi(\zeta, t)$ be a defining function of $\partial \mathrm{D}$ in $B \times \mathbb{C}_t$ and define

$$k_2(\zeta, t) = (\frac{\partial^2\varphi}{\partial\zeta\partial\overline{\zeta}}|\frac{\partial\varphi}{\partial t}|^2 - 2\Re\{\frac{\partial^2\varphi}{\partial\zeta\partial t}\frac{\partial\varphi}{\partial\zeta}\frac{\partial\varphi}{\partial t}\} + |\frac{\partial\varphi}{\partial\zeta}|^2\frac{\partial^2\varphi}{\partial t\partial\overline{t}})/|\frac{\partial\varphi}{\partial t}|^3.$$

Then, the following variation formulae for the Bergman kernels $K_{D_\zeta}(z, z)$ on the diagonal can be obtained.

Theorem 0.1[8] It holds for $(\zeta, z) \in \mathrm{D}$

$$\frac{\partial^2 K_{D_\zeta}(z, z)}{\partial\zeta\partial\overline{\zeta}} = \frac{1}{4}\int_{\partial D_\zeta}k_2(\zeta, t)(|\mathrm{L}(\zeta, t, z)|^2 + |\mathrm{K}(\zeta, t, z)|^2)ds_t$$
$$+ \iint_{D_\zeta}(|\frac{\partial\mathrm{L}(\zeta, t, z)}{\partial\overline{\zeta}}|^2 + |\frac{\partial\mathrm{K}(\zeta, t, z)}{\partial\overline{\zeta}}|^2)dxdy.$$

Using Theorem 0.1, the following two properties of variation of the Bergman kernels on the diagonal can be obtained.

Theorem 0.2 [8] Let D be a pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary, then $\log K_{D_\zeta}(z, z)$ is plurisubharmonic on D.

Theorem 0.3[8] Let D be a pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary. If, for each $\zeta \in B$, ∂D has at least one strictly pseudoconvex point, then $\log K_{D_\zeta}(z,z)$ is a strictly plurisubharmonicfunction on D.

In 2006, B. Berndtsson generalized the Theorem 0.2 to higher dimension and proved that,

Theorem 0.4[2] Let D be a pseudoconvex domain in $C_\zeta^k \times C_z^n$ and ϕ be a plurisubharmonic function on D. For each ζ let D_ζ denote the n-dimensional slice $D_\zeta := \{z \in C^n \mid (\zeta, z) \in D\}$ and by ϕ^ζ the restriction of ϕ to D_ζ. Let $K_{D_\zeta}(z,z)$ be the Bergman kernels of Bergman space $A_\zeta^2(D_\zeta, e^{-\phi^\zeta})$. Then, the function $\log K_{D_\zeta}(z,z)$ is plurisubharmonicor identically equal to $-\infty$ on D.

These Theorems give us little information about the boundary behavior of the Bergman kernels on the diagonal, but, for any invariant metric on a domain $\Omega \subset C^n$, an important characteristic is its boundary behavior. We now study the boundary behavior of the Levi form for the logarithm of Bergman kernels with respect to the parameter ζ by using complete explicit formulae of Bergman kernels on certain families. This enables us to see precisely how the Bergman kernels depend on the parameter ζ.

Firstly, we consider parameter domains with smooth boundaries which are family of discs. The considered family of discs is

$$C = \bigcup_{\zeta \in B} \{\zeta\} \times C_\zeta \text{ where } B = D_\zeta,$$

$$C_\zeta = \{z \in C_z \mid \| z - e^{i\theta(\zeta)} \| < 1\}.$$

Here, $\theta(\zeta)$ is a real-valued analytic function and $\theta(0) = 0$.

We have the following result on this pseudoconvex domain.

Theorem 1. The Levi form of $\log K_{C_\zeta}(z,z)$ with respect to ζ approaches to ∞ when $(\zeta, z) \in C$ tends to $\partial C \setminus \{(0,0)\}$ but depends on $\tan arg z$ when $(\zeta, z) \in C$ tends to $(0,0)$ in a nontrivial way. When (ζ, z) tends to $(0,0)$ in a nontrivial way, if $\tan \arg z$ tends to ∞ then $\partial^2 \log K_{C_\zeta}(z,z)/\partial \zeta \partial \overline{\zeta}$ approaches to ∞, otherwise, $\partial^2 \log K_{C_\zeta}(z,z)/\partial \zeta \partial \overline{\zeta}$ approaches to a positive non-zero constant which depends on $\tan \arg z$.

Here, the point (ζ, z) tends to the boundary in a

nontrivial way means that ζ tends to a fixed point firstly, then z tends to the boundary of the base domain A_ζ. By parity of reasoning, we will repeat no more later.

Secondly, we investigate the boundary behavior of the Bergman kernels on particular pseudoconvex domain with non-smooth boundary. Since the variation formulae in Theorem 0.1 does not make sense in the boundary when ∂D is not smooth, it is natural to ask what will happen to the Bergman kernels on the diagonal in this case. We shall give an answer to this question in a family of discs with slits.

The considered family of discs with slitsis

$$D_\zeta = \{z \in D_z \mid z \neq s\zeta, s \geq 1\},$$

where $\zeta \in B$ with $B = \{\zeta \in \mathbb{C} \mid | \zeta - 1 | < \delta, | \zeta | < 1\}$ and define

$$D = \bigcup_{\zeta \in B} \{\zeta\} \times D_\zeta.$$

Then the following result holds.

Theorem 2. The Levi form of $\log K_{D_\zeta}(z,z)$ with respect to ζ approaches to ∞ when $(\zeta, z) \in D$ tends to $(1,1) \in \partial D$ in a nontrivial way and approaches to 0 when (ζ, z) tends to $(1, \pm i) \in \partial D$ in a nontrivial way. Otherwise, $\partial^2 \log K_{C_\zeta}(z,z)/\partial \zeta \partial \overline{\zeta}$ approaches to a positive non-zero constant.

II. PRELIMINARIES

We briefly present here certain results underlying the proofs of Theorems. This exposition is adapted to our special cases.

The Bergman kernel of a domain $\Omega \subset \mathbb{C}^n$ is a reproducing kernel for the Hilbert space of all square integrable holomorphicfunctions on Ω. In what follows, let Ω be a boundeddomain in $\mathbb{C}^n$, let $A^2(\Omega)$ be the space of square integrable holomorphic functions on Ω. And let $\{\phi_j(z)\}_{j=0}^\infty$ be a complete orthonormal basis for $A^2(\Omega)$. Then the Bergman kernel $K_\Omega(z,w)$ is identified with the following series:

$$K_\Omega(z,w) = \sum_{j=0}^\infty \phi_j(z)\overline{\phi_j(w)},$$

which is independent of the choice of orthonormal basis. For $z = w$, one has

$$K_\Omega(z,z) > 0.$$

The Bergman kernel satisfies the following transformation formula.

Proposition 2.1. Let $f : \Omega^{\text{TM}} D$ be abiholomorphic

mapping between Ω and D. Then,

$$K_{\Omega}(z,w) = K_D(f(z), f(w)) \det f'(z) \overline{\det f'(w)}.$$

By Cauchy's estimate it is easy to see that $K_{\Omega}(z,w)$ is a C^{∞} function on $\Omega \times \Omega$ and on the diagonal, it can bere presented as

$$K_{\Omega}(z,z) = \sup\{| f(z) |^2 \,|\, f \in A^2(\Omega), \| f(z) \|_{A^2(\Omega)} = 1\} \text{ for } \forall\, z \in \Omega.$$

III. PROOFS OF THEOREMS

We next accomplish the proofs of Theorems that are given in the introduction.

A. Case of the Family of Discs

Proof the theorem 1.

Using the Proposition 2.1, we can get the Bergman kernels

$$K_{C_{\zeta}}(z,z) = \frac{1}{\pi \left(1- | z + e^{i\theta(\zeta)} |^2\right)^2}.$$

So,

$$\frac{\partial^2 \log K_{C_{\zeta}}(z,z)}{\partial \zeta \partial \bar{\zeta}} = 4 | \theta_{\zeta} |^2 \frac{| z |^2 \left(\Re(ze^{-i\theta(\zeta)})+2\right)}{\left(| z |^2 + 2\Re(ze^{-i\theta(\zeta)})\right)^2}.$$

Moreover, the condition $\theta(0) = 0$ induces that

$$\lim_{\zeta \to 0} \frac{\partial^2 \log K_{C_{\zeta}}(z,z)}{\partial \zeta \partial \bar{\zeta}} = 4 | \theta_{\zeta} |^2 \frac{| z |^2 \,(\operatorname{Re}z + 2)}{\left(1- | z + 1 |^2\right)^2}. \tag{1}$$

Then, from (1), if $(\zeta, z) \in C$ tends to $\partial C \setminus \{(0,0), (0,-2)\}$ then $\partial^2 \log K_{C_{\zeta}}(z,z) / \partial \zeta \partial \bar{\zeta}$ tends to ∞. Also, from (1) we have

$$\lim_{z \to -2} \lim_{\zeta \to 0} \frac{\partial^2 \log K_{C_{\zeta}}(z,z)}{\partial \zeta \partial \bar{\zeta}} = \lim_{z \to -2} \lim_{\zeta \to 0} 4 | \theta_{\zeta} |^2 \frac{1}{\left(2 + \operatorname{Re}z\right)^2} = \infty.$$

That is, if $(\zeta, z) \in C$ tends to $(0,-2) \in \partial C$ then $\partial^2 \log K_{C_{\zeta}}(z,z) / \partial \zeta \partial \bar{\zeta}$ tends to ∞ with order 1.

Finally, we consider the boundary point $(0,0)$.

Let $z = x + iy$, then

$$\lim_{\zeta \to 0} \frac{\partial^2 \log K_{C_{\zeta}}(z,z)}{\partial \zeta \partial \bar{\zeta}} = 4 | \theta_{\zeta} |^2 \frac{(x+2)}{\left(x^2 + y^2 + 4x + \dfrac{4}{(1+(y/x)^2)}\right)}.$$

Therefore, when (ζ, z) tends to $(0,0)$, the Levi form of $\log K_{C_{\zeta}}(z,z)$ with respect to ζ is dependent on $\tan \arg z$ and if $\tan \arg z$ tends to ∞ the Levi form of $\log K_{C_{\zeta}}(z,z)$ with respect to ζ tends to ∞, otherwise, the Levi form of $\log K_{C_{\zeta}}(z,z)$ with respect to ζ tends to a positive non-zero constant which depends on $\tan \arg z$.

B. Case of the Family of Discs with Slits

We will now proceed with the proof of Theorem 2.

Proof the theorem 2. Define that

$$z = E_{\zeta}(w) := e^{-i\theta} K^{-1}(e^{-t} K(w)),$$

where $K(w) = w / (1+w)^2$ is the Koebe function, $\zeta = e^{-i\theta} K^{-1}(e^{-t} / 4)$ with $t > 0, \theta \in [0, 2\pi)$, this is inspired by [1].

For each ζ, $E_{\zeta}(w)$ maps the unit disc D in the w plane to D_{ζ} which is the unit disc in the complex z-plane minus a segment $[\zeta, e^{-i\theta})$.

Then the inverse mapping of $E_{\zeta}(w)$ is

$$w = E_{\zeta}^{-1}(z) = K^{-1}(e^t K(e^{i\theta} z)).$$

We now can get the power series expansion of $E_{\zeta}^{-1}(z)$ with respect to the parameter ζ in a neighborhood of $\zeta = 1$, that is, in a neighborhood of $(t, \theta) = (0,0)$ which is

$$E_{\zeta}^{-1}(z) = K^{-1}(e^t K(e^{i\theta} z))$$

$$= z + \frac{K(z)}{K'(z)}t + iz\theta + \left(\frac{K(z)}{K'(z)} - \frac{1}{2}\frac{K(z)^2 K''(z)}{(K'(z))^3}\right)t^2$$

$$- \frac{1}{2}z\theta^2 + \frac{1}{2}\left(1 - \frac{K(z)}{K'(z)}\frac{K''(z)}{K'(z)}\right)izt\theta + O(t^3) + O(\theta^3)$$

$$= z + \frac{z(1+z)}{(1-z)}t + iz\theta + \frac{1}{2}\frac{z(1+z)(1+2z-z^2)}{(1-z)^3}t^2$$

$$- \frac{1}{2}z\theta^2 + \frac{1}{2}\frac{1+2z-z^2}{(1-z)^2}izt\theta + O(t^3) + O(\theta^3).$$

On the other hand, as it is well known, the Bergman kernel of the unit disc D on the diagonal is

$$K_D(w,w) = \frac{1}{\pi}\frac{1}{(1- | w |^2)^2}.$$

Then, by Proposition 2.1, the Bergman kernels of D_{ζ} on the diagonal are given by

$$K_{D_{\zeta}}(z,z) = \frac{1}{\pi}\frac{1}{(1- | E_{\zeta}^{-1}(z) |^2)^2} | (E_{\zeta}^{-1}(z))_z |^2 .$$

Furthermore, we can calculate that

$$\lim_{\zeta \to 1} \frac{\partial^2 \log(1 - |E_\zeta^{-1}(z)|^2)}{\partial \zeta \partial \overline{\zeta}} = \frac{-2|z|^2 \operatorname{Re}((1+z)/(1-z))}{(1-|z|^2)},$$

$$\lim_{\zeta \to 1} \frac{\partial^2 \log |(E_\zeta^{-1}(z))_z|^2}{\partial \zeta \partial \overline{\zeta}} = \frac{1}{4} \operatorname{Re}\left(\frac{1-3z^2}{(1-z)^2}\right).$$

Now let $z = re^{i\theta} \in D_\zeta$, we conclude that

$$\lim_{\zeta \to 1} \frac{\partial^2 \log K_{D_\zeta}(z,z)}{\partial \zeta \partial \overline{\zeta}} = \frac{1}{4} \frac{1 + r^2 \cos 2\theta}{1 + r^2 - 2r \cos \theta}.$$

Then

$$\lim_{r \to 1} \lim_{\zeta \to 1} \frac{\partial^2 \log K_{D_\zeta}(z,z)}{\partial \zeta \partial \overline{\zeta}} = \frac{1}{4}\left((1 - \cos\theta) + \frac{1}{(1 - \cos\theta)} - 2\right). \tag{2}$$

If $\zeta \to 1$ then the singular point of the boundary D_ζ tends to $z = 1$. We see from (2) that

(1) $\partial^2 \log K_{D_\zeta}(z,z)/\partial \zeta \partial \overline{\zeta}$ tends to ∞ with order 2 as $(\zeta, z) \in D$ tends to $(1,1)$.

(2) $\partial^2 \log K_{D_\zeta}(z,z)/\partial \zeta \partial \overline{\zeta}$ tends to 0 with order 2 as $(\zeta, z) \in D$ tends to $(1, \pm i)$.

(3) $\partial^2 \log K_{D_\zeta}(z,z)/\partial \zeta \partial \overline{\zeta}$ tends to a positive number as $(\zeta, z) \in D$ tends to $(1, z) \in \partial D$ here $z \neq 1, \pm i$.

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